Ch 3: Inverse function.

And its calculus

§3.1 : Exponential function

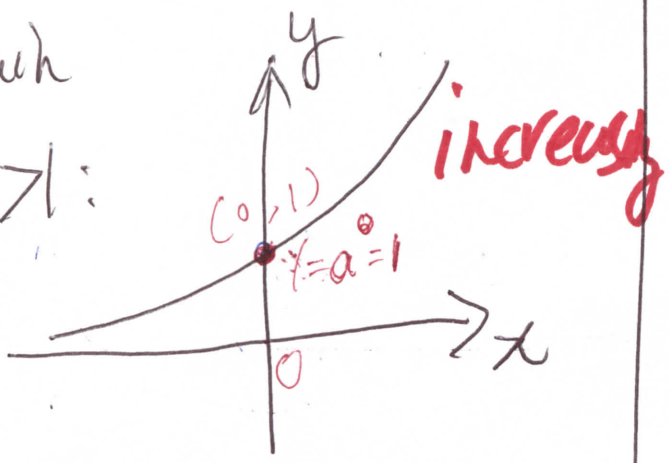
Inverse function

§3.2 : Logarithmic function

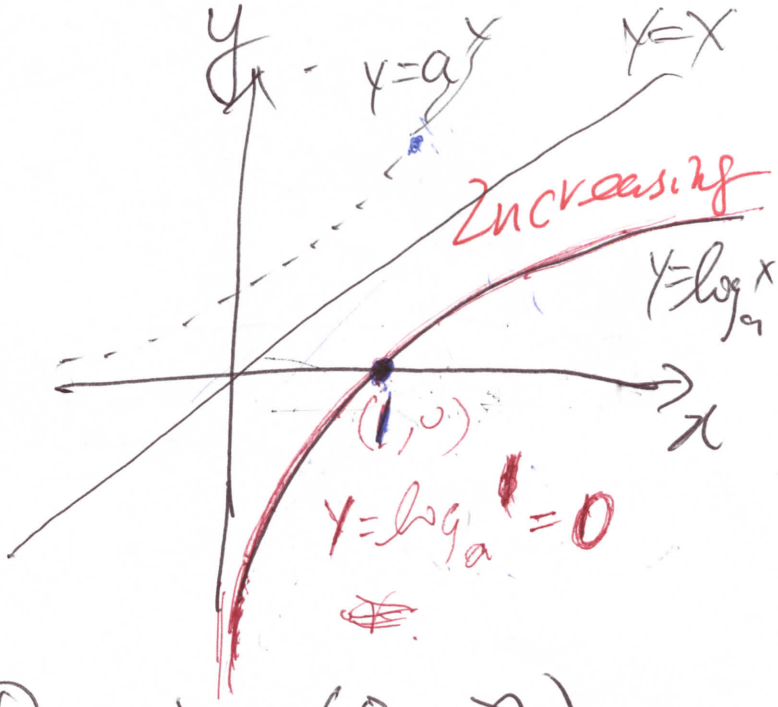
$y = a^x$ ($a>1$ or $0<a<1$)	$y = \log_a x$ ($a>1$)

Domain

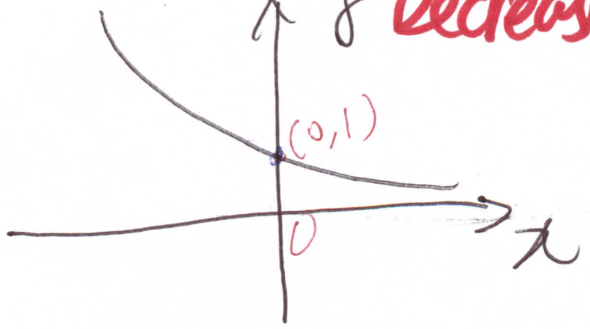
$a>1$:



increasing



$0<a<1$:



Decreasing.

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

Their graphs are symmetric about $y=x$.

$y = a^x$

$a > 1$: $\lim_{x \to \infty} a^x = \infty$

$\lim_{x \to -\infty} a^x = 0$

$0 < a < 1$: $\lim_{x \to \infty} a^x = 0$

$\lim_{x \to -\infty} a^x = \infty$

$\Rightarrow$ H.A : $y = 0$ (x-axis)

$y = \log_a x$ $(a > 1)$

$\lim_{x \to \infty} \log_a x = \infty$

$\lim_{x \to 0^+} \log_a x = -\infty$

$\Rightarrow$ V.A : $x = 0$ (y-axis)

laws of exponential

$a^{x+y} = a^x \cdot a^y$

$a^{x-y} = \frac{a^x}{a^y}$

$(a^x)^y = a^{xy}$

$(ab)^x = a^x b^x$

Laws of logarithm

$\log_a (xy) = \log_a x + \log_a y$

$\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^r = r \log_a x$

The natural number "e" is defined as

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

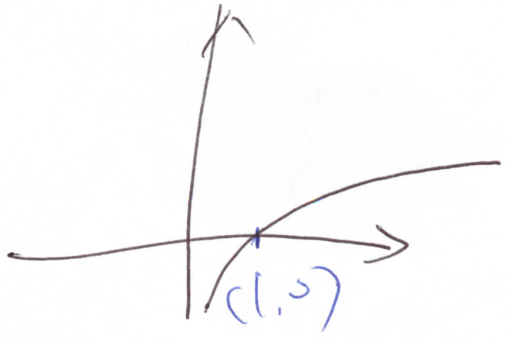
1^∞ indeterminate

"Euler number"

$$e \approx 2.71828\ldots$$

an irrational number.

Another name: Euler's number

$y = e^x$	$y = \log_e x = \ln x$
natural exponential	natural logarithm
	

$$e^{\ln x} = x$$

$$\ln e^{x} = x$$

Cancellation

or

$$a^{\log_a x} = x$$

works!!

$$\log_a a^{x} = x$$

$$\log_a a^{x} = x \log_a a = x \text{ because}$$

$$\log_a a = 1 \iff a^1 = a$$

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

$= \square$

Remark:

According to the other property of logarithm:

"change of Base-formula":

$$\log_a x = \frac{\ln x}{\ln a} = \frac{\log_e x}{\log_e a}$$

Proof:

Let $y = \log_a x$

By cancellation ⇓ property:

$$a^y = a^{\log_a x} = x$$

⇓

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$\log_a x = y = \frac{\ln x}{\ln a}$$ #

Ex3: Evaluate

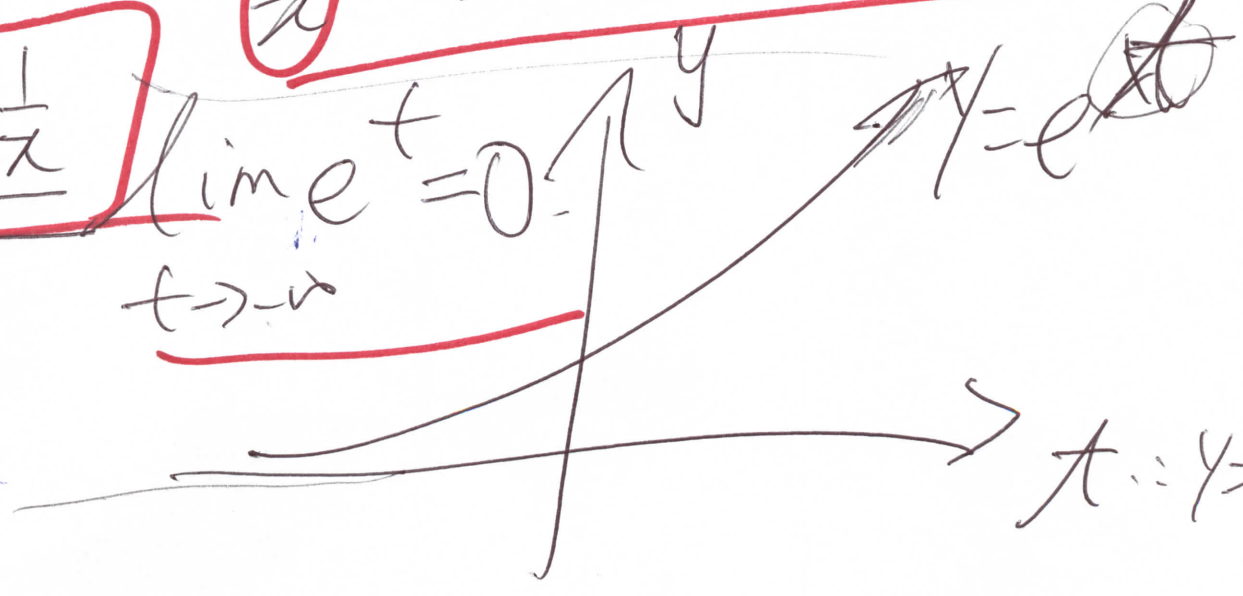
$$\lim_{x \to 0^-} e^{\frac{1}{x}}$$

Solution: $x \to 0^- \Rightarrow \frac{1}{x} \to -\infty$.

$$\Rightarrow \lim_{\frac{1}{x} \to -\infty} e^{\frac{1}{x}} = 0.$$

$t = \frac{1}{x}$

$$\lim_{t \to -\infty} e^t = 0$$



Ex 4:

$$\lim_{x \to \infty} \frac{e^{2x}}{e^{2x}+1}$$

$$= \lim_{x \to \infty} \frac{\frac{e^{2x}}{e^{2x}}}{\frac{e^{2x}+1}{e^{2x}}}$$

$$= \lim_{x \to \infty} \frac{1}{1+e^{-2x}}$$

$$= \frac{1}{1+\lim_{x \to \infty} e^{-2x}} = \frac{1}{1+\lim_{-2x \to -\infty} e^{-2x}}$$

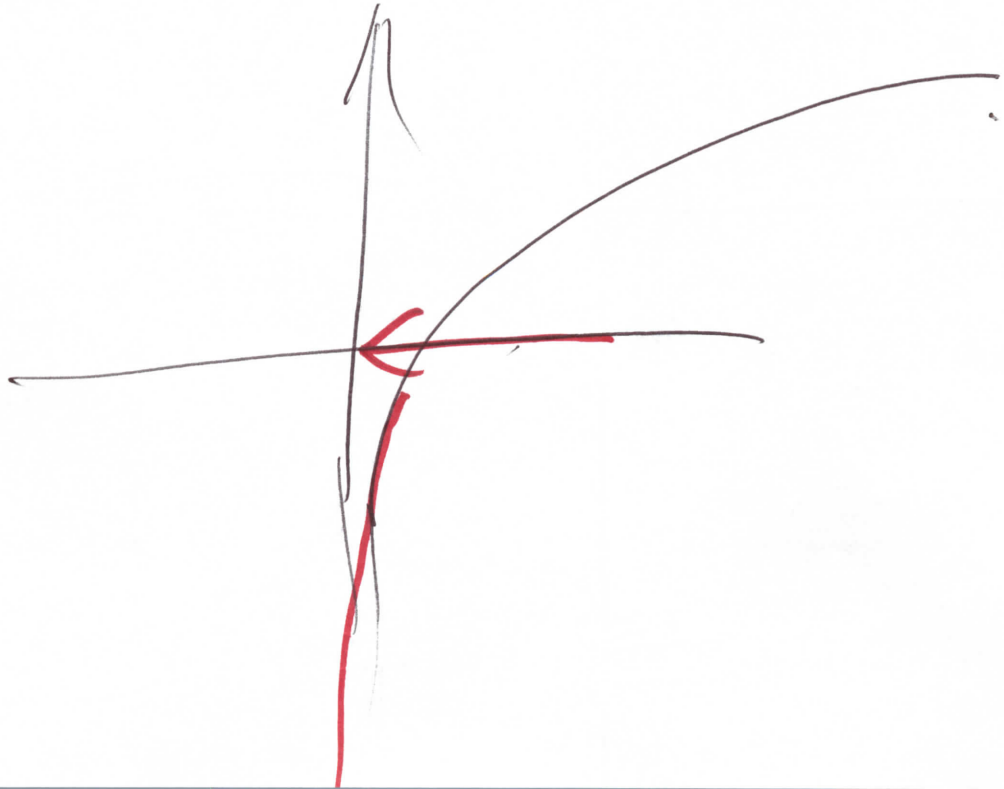
$$= \frac{1}{1+0} = 1 \quad \#$$

Ex: $\lim_{x\to 0} \log_{10}(\tan^2 x)$.

Solution: $x \to 0 \Rightarrow \tan x \to \tan 0 = 0$

$\Rightarrow \tan^2 x \to 0^+$

$\lim_{\tan^2 x \to 0^+} \log_{10}(\tan^2 x) = -\infty$. #



$$\lim_{x\to\infty}\left(\ln(1+x^2)-\ln(1+x)\right)$$

$$=\lim_{x\to\infty}\ln\frac{1+x^2}{1+x}$$

$$=\ln\left(\lim_{x\to\infty}\frac{1+x^2}{1+x}\right)$$ (Because Continuity)

$$=\ln\left(\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{x^2}{x}}{\frac{1}{x}+\frac{x}{x}}\right)$$

$$=\ln\left(\lim_{x\to\infty}\frac{\frac{1}{x}+x}{\frac{1}{x}+1}\right)$$

$$=\ln\left(\frac{\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}x}{\lim_{x\to\infty}\frac{1}{x}+1}\right)$$ ($\lim\frac{1}{x}\to 0$, $\lim x\to\infty$, $\lim\frac{1}{x}\to 0$)

$$=\ln\frac{\infty}{1}=\ln\infty=\infty$$ #

ch3: Inverse functions.

§3.2 Inverse function and logarithms

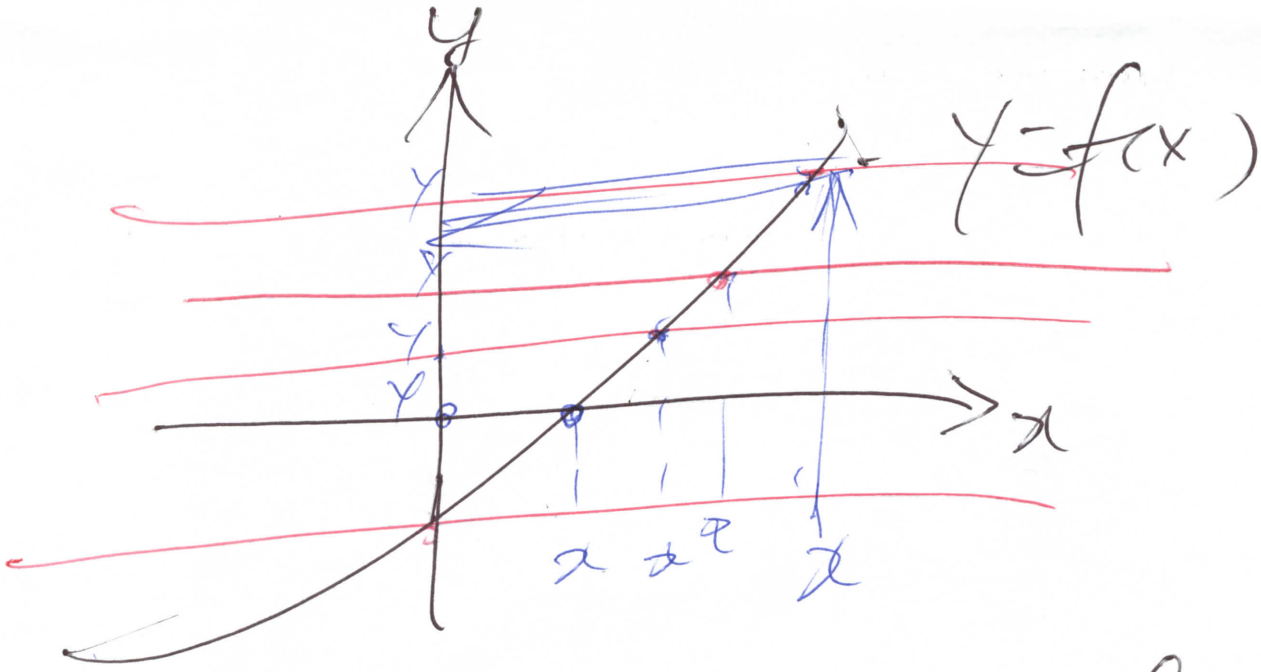
1: Definition of "one-to-one" func.

A function $f(x)$ is called one-to-one function if it never takes on the same value twice. That is:

$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

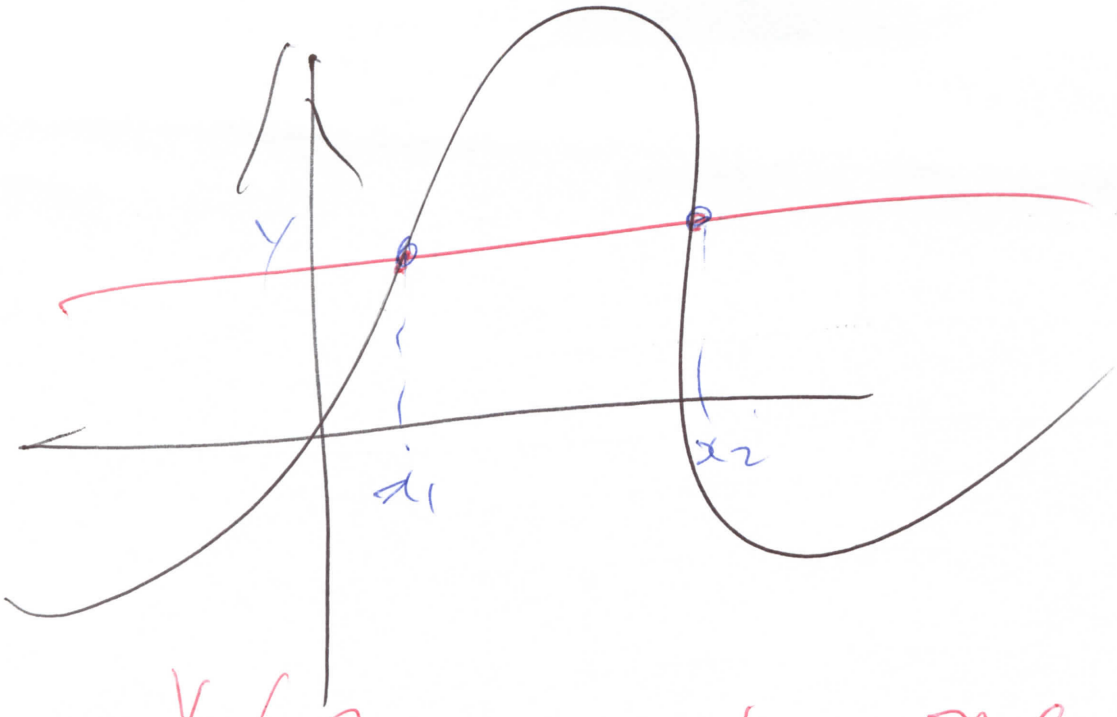
$\Rightarrow$ Different "x" $\Rightarrow$ different "y"

2: Horizontal line Test.



A function is one-to-one function if and only if no horizontal line intersects its graph more than once.

$\Longleftrightarrow$ Only one intersection point between the horizontal line and the graph.



Not one-to-one!!!

3: Definition of inverse function:

Let $f(x)$ be a one-to-one function with domain A and range B. Then its inverse function, $\boxed{f^{-1}}$, has the domain B and the range A.

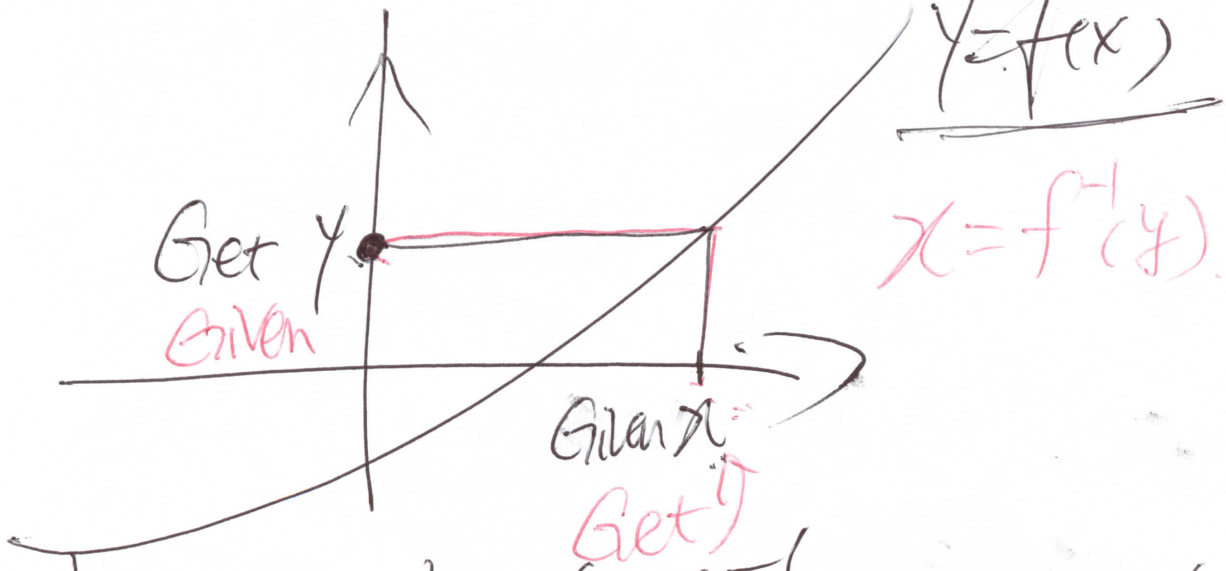
Remarks: (1) Only one-to-one function has its inverse function

$y = f(x)$ $\overset{\text{one-to-one}}{\Longrightarrow}$ $x = f^{-1}(y)$.

Domain : A Domain : B

Range : B Range : A

(2)



(3) The graph of f^{-1} is symmetric with the graph of f about the line $\boxed{y = x}$

④ Cancellation properties

$$f(f^{-1}(x)) = x, \quad x \in B$$

or

$$f^{-1}(f(x)) = x \therefore x \in A$$

⑤ How to find $y = f^{-1}(x)$?

Ex1: Given $f(x) = x^3 + 2$, find $f^{-1}(x)$.

Solution: Step ①: Write $y = f(x)$

$$\Rightarrow y = x^3 + 2$$

Step ②: Solve this equation for x in terms of y

$\Rightarrow \sqrt[3]{x^3} = \sqrt[3]{y-2} \Rightarrow \boxed{x = \sqrt[3]{y-2}}$

Step 3: To express f^{-1} as a function of x, interchange x and y.

$\Rightarrow y = f^{-1}(x)$

$\Rightarrow \boxed{y = \sqrt[3]{x-2}}$

interchange x and y.

4: The calculus of inverse functions

① Theorem: If $f(x)$ is one-to-one function and continuous, defined on an interval, then its inverse function $f^{-1}(x)$ is also continuous.

② Theorem: If $f(x)$ is one-to-one differentiable function with inverse function $f^{-1}(x)$, and $f'(f^{-1}(a)) \neq 0$, then the inverse function $f^{-1}(x)$ is differentiable at "a" and: $(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$.

Proof:

Remarks: (1) From the given formula, we know "a" is in the domain of $f^{-1} \Rightarrow f^{-1}(a)$ is defined.

$\Rightarrow a$ is in the range of $f(x)$

$\Downarrow$

there exists a number "x" such that

$$\boxed{f(x) = a \Rightarrow x = f^{-1}(a)}$$

(2) : $f' = \frac{dy}{dx} \Rightarrow \frac{1}{f'} = \frac{dy}{dx}$

$$\frac{1}{f'} = \frac{dx}{dy} = (f^{-1})'$$

$$\Rightarrow \boxed{(f^{-1})' = \frac{1}{f'} \Leftrightarrow \frac{dx}{dx} = \frac{1}{\frac{dx}{dy}}}$$

Ex 2: Given $f(x) = 2x + \cos x$,

find $(f^{-1})'(1)$

Solution: $a = 1 \Rightarrow f^{-1}(a) = f^{-1}(1)$

$\Rightarrow$ At what value of x,

$f^{-1}(f(x)) = f^{-1}(1) \Rightarrow x = f^{-1}(1)$

Solve the equation for x:

$\Rightarrow 2x + \cos x = 1$.

$\Rightarrow$ let $x = 0$, the

$2 \cdot 0 + \cos 0 = 0 + 1 = 1$

$\Rightarrow x = 0 \Rightarrow$ $f^{-1}(1) = 0$

Thus: $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))}$

$= \dfrac{1}{f'(0)}$

and $f'(x) = (2x + \cos x)'$

$= 2 - \sin x.$

$\Rightarrow (f^{-1})'(1) = \dfrac{1}{2 - \sin 0} = \dfrac{1}{2}$ #

§3.3 Derivatives of Logarithmic and exponential functions.

1: Theorem:

$$\boxed{(\log_a x)' = \frac{1}{x}\log_a e}$$

change the base

$$= \frac{1}{x}\frac{\ln e}{\ln a}$$

$$= \frac{1}{x\ln a}$$

Proof: Let $f(x) = \log_a x$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\log_a (x+h) - \log_a x}{h}$$

$$= \lim_{h\to 0}\frac{\log_a \frac{x+h}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\log_a\left(1+\frac{h}{x}\right)}{h} = \lim_{h\to 0} \frac{1}{h}\log_a\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{h}\log_a\left(1+\frac{h}{x}\right)^{\frac{1}{\frac{h}{x}\cdot x}} = \frac{1}{x}$$

$$= \lim_{h\to 0} \log_a\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0} \frac{1}{x}\log_a\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \lim_{h\to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$= \lim_{h\to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{x}{h}\cdot\frac{1}{x}}$$

$$= \lim_{h\to 0} \frac{1}{x} \log_a \left[\left(1+\frac{h}{x}\right)^{\frac{x}{h}}\right]^{\frac{1}{x}}$$

$$= \frac{1}{x} \lim_{h\to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{h}{x}}$$

$$= \frac{1}{x} \lim_{\frac{h}{x}\to 0} \log_a \left(1+\frac{h}{x}\right)^{\frac{1}{\frac{h}{x}}}$$

$y = \frac{h}{x}$

$$= \frac{1}{x} \lim_{y\to 0} \log_a (1+y)^{\frac{1}{y}}$$

$$= \frac{1}{x} \log_a \boxed{\lim_{y \to 0} (1+y)^{\frac{1}{y}}} = e$$

$$\Downarrow$$

Natural number

$$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$$

1^{∞}

$$= \frac{1}{x} \log_a e \qquad \#$$

Continuity: $f(x)$ is continuous

$$\lim_{x \to a} f(x) = f(a)$$

$$= f\left(\lim_{x \to a} x\right)$$

2: $(\ln x)' = \frac{1}{x \ln e} = \frac{1}{x}$.

$$\boxed{(\ln x)' = \frac{1}{x}} \quad (x \neq 0)$$

Ex1: Given $y = \ln(x^3+1)$

find $\frac{dy}{dx}$.

"chain Rule"

Solution: $\frac{d}{dx}(\ln(x^3+1))$

$$= \frac{1}{x^3+1} \cdot (x^3+1)'$$

$$= \frac{3x^2}{x^3+1} \quad \#$$

Ex2: $(\log_{10}(2+\sin x))'$

$$= \frac{1}{2+\sin x}\cdot\frac{1}{\ln 10}\cdot(2+\sin x)'$$

$$= \frac{\cos x}{(2+\sin x)\ln 10}$$ #

Ex3: Find $f'(x)$ if $f(x)=\ln|x|$ $(x\neq 0)$.

Solution: $|x| = \begin{cases} x, & \text{if } x>0 \\ -x, & \text{if } x<0 \end{cases}$

$$\Rightarrow \ln|x| = \begin{cases} \ln x, & \text{if } x>0 \\ \ln(-x), & \text{if } x<0 \end{cases}$$

$$\Rightarrow (\ln|x|)' = \begin{cases} (\ln x)' = \boxed{\frac{1}{x}}, & \text{if } x > 0 \\ (\ln(-x))' & \text{if } x < 0 \end{cases}$$

$$\| $$

$$\frac{1}{-x} \cdot (-x)'$$

$$\| $$

$$\frac{1}{-x}(-1) = \boxed{\frac{1}{x}}$$

$$\Rightarrow \boxed{(\ln|x|)' = \frac{1}{x}} \quad \ddots$$

Logarithmic Differentiation

Ex 4: Given $y = \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

find $\dfrac{dy}{dx}$.

Solution: $\ln y = \ln \dfrac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

$\Rightarrow \ln y = \ln x^{\frac{3}{4}}\sqrt{x^2+1} - \ln(3x+2)^5$

$\Rightarrow \ln y = \ln x^{\frac{3}{4}} + \ln (x^2+1)^{\frac{1}{2}} - 5\ln(3x+2)$

$\Rightarrow (\ln y)' = \left(\frac{3}{4}\ln x\right)' + \left(\frac{1}{2}\ln(x^2+1)\right)' - \left(5\ln(3x+2)\right)'$

(2) Use the laws of logarithms to simplify $\ln f(x)$

(3) Use the implicit differentiation to differentiate $\ln y = \ln f(x)$.

$(\ln y)' = \frac{1}{y}\frac{dy}{dx} = \frac{1}{f(x)} f'(x)$

(4) Solve the resulting equation for $\frac{dy}{dx}$.

$$\Rightarrow \frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2+1}\cdot 2x$$

$$- 5\frac{1}{3x+2}\cdot 3$$

$$\Rightarrow \frac{dy}{dx} = y\cdot\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$= \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

#.

Strategy of logarithmic differentiation

① Take natural logrithm of both sides of $y = f(x)$.

$$\Rightarrow \ln y = \boxed{\ln f(x)}.$$

$$y = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}$$

$$\Downarrow$$

$$\ln y = \ln \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}$$

$$\ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2+1)^2$$

$$\frac{d}{dx}\ln y = \ln \sin^2 x + \ln \tan^4 x - 2\ln(x^2+1)$$

$$\frac{d}{dx}\left[2\ln(\sin x) + 4\ln(\tan x) - 2\ln(x^2+1)\right]$$

$$\Downarrow$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{\sin x}\cos x + 4\frac{1}{\tan x}\sec^2 x - 2\cdot\frac{1}{x^2+1}\cdot 2x$$

$$\frac{dy}{dx} = y\cdot\left[2\cot x + 4\frac{1}{\cos^2 x}\cdot\frac{\cos x}{\sin x} - \frac{4x}{x^2+1}\right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left[2\cot x + \frac{4}{\sin x \cos x} - \frac{4x}{x^2+1}\right].$$

5: Derivative of exponential functions.

$$(a^x)' = \frac{d}{dx}(a^x) = \boxed{a^x \ln a}$$

Proof: Let $y = a^x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(a^x)$

$\log_a$ $\log_a$

$$\log_a y = \log_a a^x = x.$$

$$(\log_a y)' = (x)'$$

$$\frac{1}{y}\frac{1}{\ln a}\cdot\frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = y \ln a = a^x \ln a$$

$$\Rightarrow \boxed{\frac{dy}{dx} = a^x \ln a}$$

б: $\frac{d}{dx}(e^x) = e^x \ln e = e^x$

$$\Downarrow$$

$$(e^x)' = \boxed{\frac{d}{dx}(e^x) = e^x}$$

$$\Rightarrow \frac{dy}{dx} = 2 \ln 10 \times 10^{x^2} \quad \#$$

Solution 2: $\frac{d}{dx}(10^{x^2})$

$$= 10^{x^2} \ln 10 \cdot (x^2)'$$

$$= \boxed{2 \ln 10 \times 10^{x^2}} \quad \#$$

~~Note~~

Theorem: $(\log_a x)' = \frac{1}{x}\log_a e$

$$\Rightarrow (\log_a x)' = \frac{1}{x}\frac{\ln e}{\ln a} = \frac{1}{x}\frac{\log_e e}{\ln a}$$

$$\boxed{(\log_a x)' = \frac{1}{x}\frac{1}{\ln a}} \quad \boxed{= \frac{1}{x\ln a}}$$

$$\log_a a = 1$$

$$\Uparrow$$

Proof: $a^{\boxed{\log_a a}} = a = a^{\boxed{1}}$

Ex 6: $\frac{d}{dx}(10^{x^2})$

Solution: Let $y = 10^{x^2}$

we are finding $\frac{dy}{dx}$

$$\Rightarrow (\ln y)' = \ln 10^{x^2} = (x^2 \ln 10)'$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = (\ln 10)(2x)$$

$$\Rightarrow \frac{dy}{dx} = y(\ln 10)(2x)$$

$$= 10^{x^2} \ln 10\,(2x).$$

§3.4: O.D.E. (Applications)

Ordinary Differential equation

"Skipped."

§3.5: Inverse Trigonometric functions.

1: $y = \sin x \Rightarrow$

$\Downarrow$

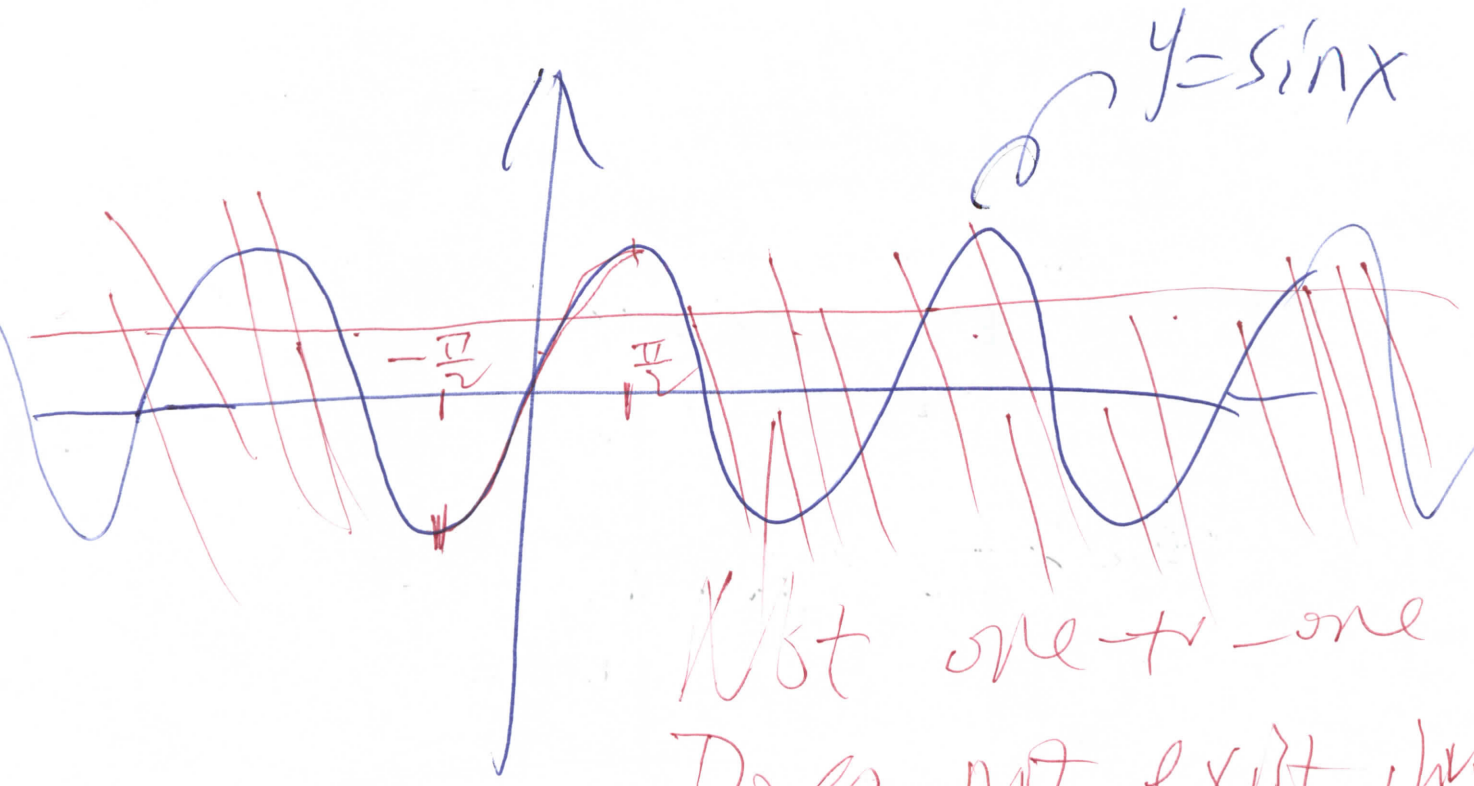
$\sin^{-1} y = \sin^{-1}(\sin x) = x.$

$\Downarrow$

$x = \sin^{-1} y.$

$\Downarrow$

$$\boxed{y = \sin^{-1} x} = \arcsin x$$



Not one-to-one

Does not exist inverse.

Domain: $(-\infty, \infty)$

monotone fine.

increasing

or decreasing.

§3.4 skipped.

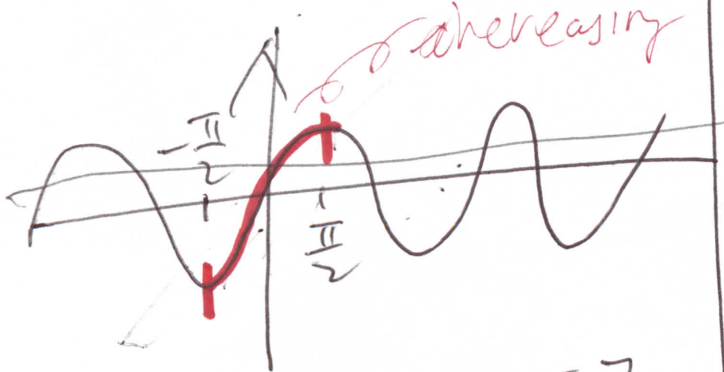
§3.6 skipped.

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§3.5 Inverse trigonometric functions

$x = \arcsin y \Leftrightarrow y = \arcsin x$

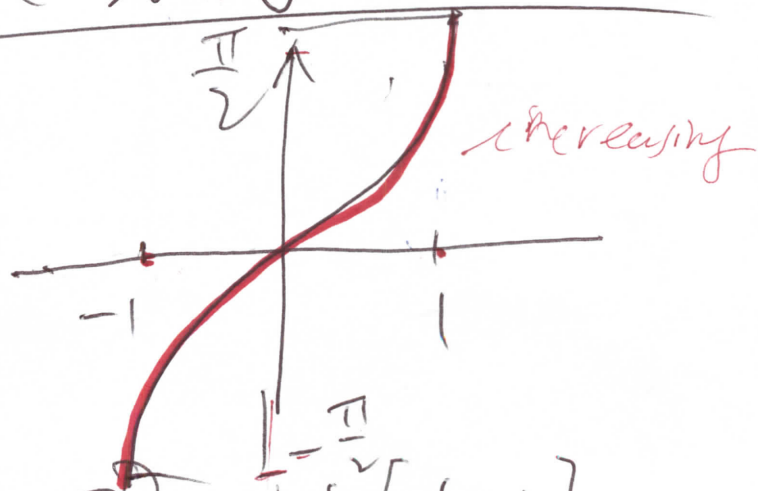
$y = \sin x$



Domain: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Range: $[-1, 1]$

$x = \sin^{-1} y \Leftrightarrow y = \sin^{-1} x$



Domain: $[-1, 1]$

Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$

② Cancellation properties:

(i) $\sin(\sin^{-1})(x) = \boxed{x}$; $x \in [-1,1]$

(ii) $\sin^{-1}(\sin x) = \boxed{x}$; $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

③ $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$.

Let $y = \sin^{-1}x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)$

$\Downarrow$

$\sin y = \sin(\sin^{-1}x) = x$

$(\sin y)' = x' = 1$

$\Downarrow$

$\cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$

Because $\sin y = x \Rightarrow \sin^2 y + \cos^2 y = 1$

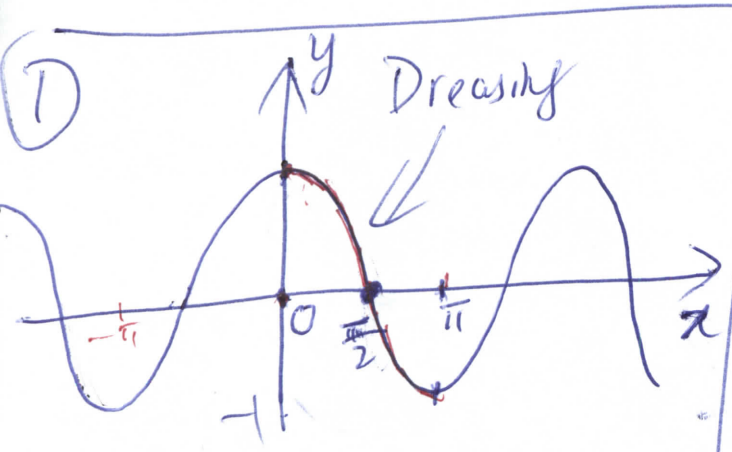
$\Rightarrow \cos^2 y = 1 - \sin^2 y = 1 - x^2 \Rightarrow \cos y = \pm\sqrt{1-x^2}$

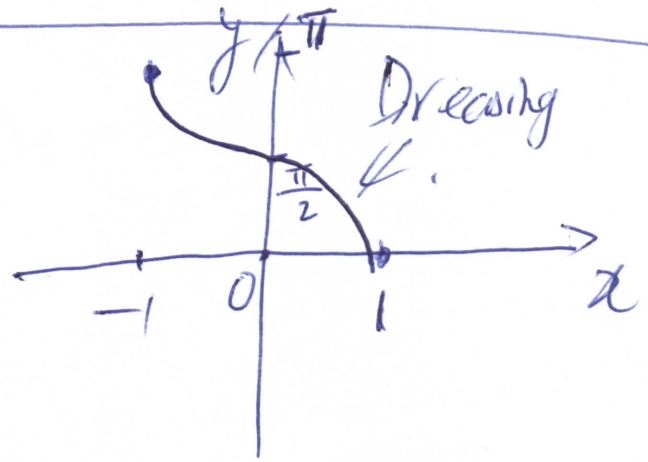
further because $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, thus $\cos y = \sqrt{1-x^2}$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$y = \cos x$

$x = \cos^{-1} y \iff y = \cos^{-1} x$

$x = \arccos y \iff y = \arccos x$

① 



Domain: $[0, \pi]$

Range: $[-1, 1]$

Domain: $[-1, 1]$

Range: $[0, \pi]$

② Cancellation property:

$\cos(\cos^{-1} x) = x$, for $x \in [-1, 1]$

$\cos^{-1}(\cos x) = x$, for $x \in [0, \pi]$

(3) $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$

Let $y = \cos^{-1}x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\cos^{-1}x)$

$\Downarrow$

$\cos y = \cos(\cos^{-1}x) = x$

$\Downarrow$

$\boxed{\cos y = x}$

$\Downarrow$

$(\cos y)' = x' = 1$

$\Downarrow$

$-\sin y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = -\frac{1}{\sin y}$.

By $\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \pm\sqrt{1-\cos^2 y} = \pm\sqrt{1-x^2}$

Because $y = \cos^{-1}x \in [0, \pi] \Rightarrow \sin y = \sqrt{1-x^2}$

$\Rightarrow \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$

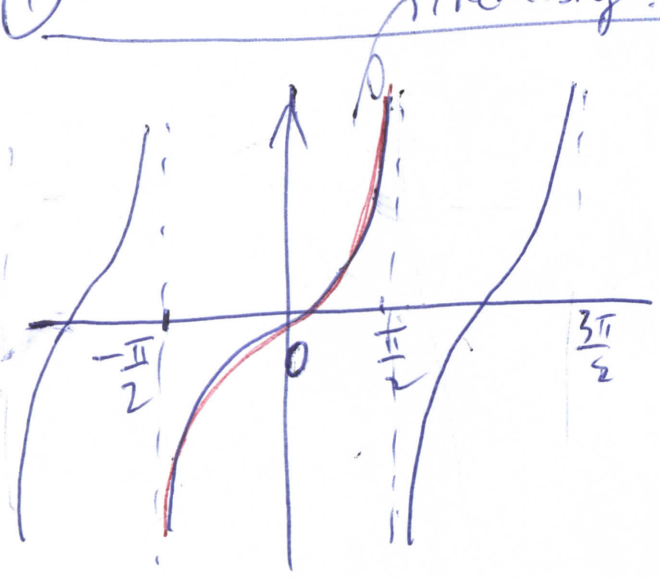
$y = \tan x$

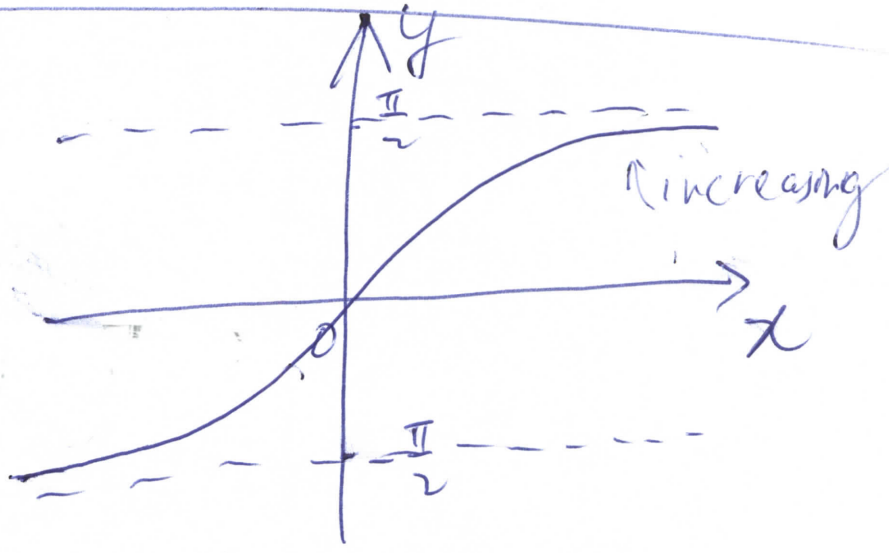
$x = \tan^{-1} y \iff y = \tan^{-1} x$

or

$x = \arctan y \iff y = \arctan x$.

① increasing.





Domain: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Range: $(-\infty, \infty)$

Vertical asymptotes:

$x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$

Domain: $(-\infty, \infty)$

Range: $(-\frac{\pi}{2}, \frac{\pi}{2})$

Horizontal asymptotes:

$y = \frac{\pi}{2}$, $y = -\frac{\pi}{2}$.

$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$

$\lim_{x \to -\infty} \tan^{-1} x = -\frac{\pi}{2}$

② Cancellation properties:

$\tan(\tan^{-1} x) = x$, for $x \in (-\infty, \infty)$

$\tan^{-1}(\tan x) = x$, for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

(3) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$.

Let $y = \tan^{-1}x \Rightarrow \frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}x)$.

$\Downarrow$

$\tan y = \tan(\tan^{-1}x) = x$.

$\Downarrow$

$\boxed{\tan y = x}$

$\Downarrow$

$(\tan y)' = x' = 1$

$\Downarrow$

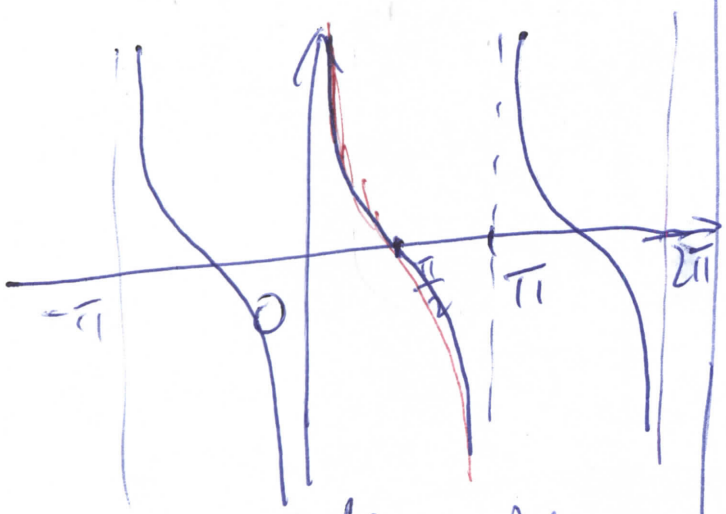
$\sec^2 y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

Because $1 + \tan^2 y = \sec^2 y \Rightarrow \sec^2 y = 1 + x^2$

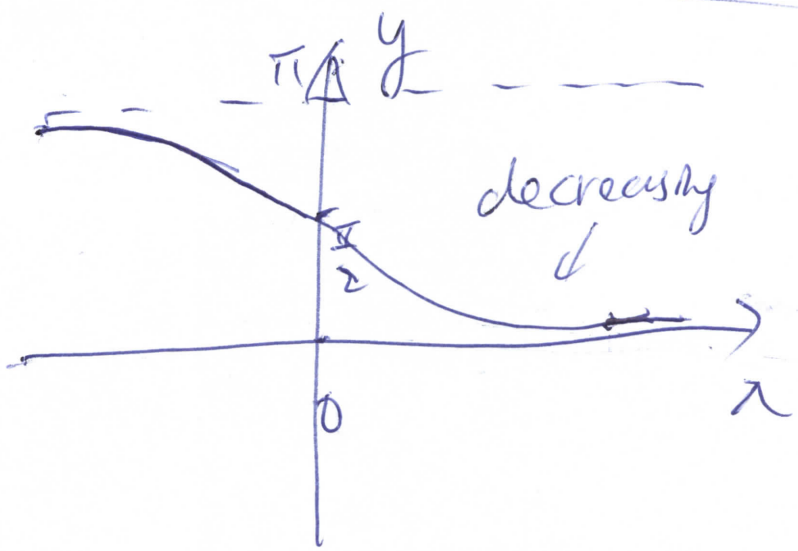
Thus: $\frac{dy}{dx} = \frac{1}{1+x^2}$ #

$y = \cot x$

$x = \cot^{-1} y \iff y = \cot^{-1} x$
or
$x = \operatorname{arc\,cot} y \iff y = \operatorname{arc\,cot} x$



decreasing



Domain: $(0, \pi)$

Range: $(-\infty, \infty)$

V.A.:
$x = 0$, and $x = \pi$

Domain: $(-\infty, \infty)$

Range: $(0, \pi)$

H.A.:
$y = 0$, and $y = \pi$

$$\lim_{x \to \infty} \cot^{-1} x = 0$$

$$\lim_{x \to -\infty} \cot^{-1} x = \pi$$

$$\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

Ex1: Simplify $\cos(\tan^{-1}x)$.

Solution: let $y = \tan^{-1}x$

$\Downarrow$

$\tan y = \tan(\tan^{-1}x) = x$

$\Downarrow$

$x = \tan y$

We are simplifying "$\cos y$" now

Because $\cos y = \frac{1}{\sec y}$, and $\sec^2 y = 1 + \tan^2 y$.

$\Rightarrow \sec^2 y = 1 + x^2 \Rightarrow \sec y = \pm\sqrt{1+x^2}$

Because $y = \tan^{-1}x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow \sec y = \sqrt{1+x^2}$.

$\Rightarrow \cos y = \frac{1}{\sqrt{1+x^2}} \Rightarrow \cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$ #

Ex2: Evaluate

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right)$$

Solution: $x \to 2^+ \Rightarrow x-2 \to 0^+$

$$\Rightarrow \frac{1}{x-2} \longrightarrow \infty.$$

$$\Rightarrow \arctan\left(\frac{1}{x-2}\right) \longrightarrow \frac{\pi}{2}.$$

$$\Rightarrow \lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \frac{\pi}{2}$$

#

$$\lim_{x \to 2^-} \arctan\left(\frac{1}{x-2}\right) = -\frac{\pi}{2}$$

Ex3: Differentiate $f(x) = x \tan^{-1}\sqrt{x}$.

Solution: $f'(x) = x' \tan^{-1}\sqrt{x} + x(\tan^{-1}\sqrt{x})'$

$$= \tan^{-1}\sqrt{x} + x \frac{1}{1+(\sqrt{x})^2} \cdot (\sqrt{x})'$$

$$= \tan^{-1}\sqrt{x} + \frac{x}{1+x} \frac{1}{2\sqrt{x}}$$

$$= \tan^{-1}\sqrt{x} + \frac{\sqrt{x}}{2(1+x)} \quad \#.$$

§3.6: Hyperbolic functions

Skipped.

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \left(\text{hyperbolic } \underset{\text{sin}}{\sin}\right)$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \left(\text{hyperbolic } \underset{\text{cos}}{\cos}\right)$$

$$\tanh x = \frac{\sinh}{\cosh} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh}{\sinh} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$\operatorname{csch} x = \frac{1}{\sinh}$$

$$\operatorname{sech} x = \frac{1}{\cosh}$$

§3.7: Indeterminate forms and L'Hospital's Rule.

1: If:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{0}{0}.$$

or:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}=\frac{\pm\infty}{\pm\infty}.$$

both are called indeterminate.

(Quotient) form of limit, and, they may exist or may not exist.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

$$\Downarrow$$

$$\frac{0}{0}$$

2. L'Hospital's Rule.

Suppose $f(x)$ and $g(x)$ are differentiable and $g'(x) \neq 0$ near the number "a".

Suppose $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$.

or $\lim_{x \to a} f(x) = \pm\infty$, $\lim_{x \to a} g(x) = \pm\infty$

then: $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

"$\frac{0}{0}$" or "$\frac{\infty}{\infty}$"

finding limit

Proof: Skipped.

Remark: L'Hospital's Rule works for

Ex1: $\lim_{x\to 1} \frac{\ln x}{x-1}$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

Prof: ① $\lim_{x\to 0} \frac{\sin x}{x} = \frac{\lim_{x\to 0} \sin x}{\lim_{x\to 0} x} = \frac{0}{0}$

② Apply L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{(\sin x)'}{x'} = \lim_{x\to 0} \frac{\cos x}{1}$$

$$= \lim_{x\to 0} \cos x = \cos 0 = 1 \quad \#$$

Ex1: $\lim_{x\to 1}\frac{\ln x}{x-1}=\frac{\lim_{x\to 1}\ln x}{\lim_{x\to 1}(x-1)}=\frac{0}{0}$

$\Rightarrow \lim_{x\to 1}\frac{(\ln x)'}{(x-1)'}=\lim_{x\to 1}\frac{\frac{1}{x}}{1}$

$=\lim_{x\to 1}\frac{1}{x}=1$ #

Ex3: $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}$

Solution: Because $\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\frac{\lim_{x\to\infty}\ln x}{\lim_{x\to\infty}\sqrt[3]{x}}$

$=\frac{\infty}{\infty}$: indeterminate form

$\lim_{x\to\infty}\frac{\ln x}{\sqrt[3]{x}}=\lim_{x\to\infty}\frac{(\ln x)'}{(x^{\frac{1}{3}})'}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{3}x^{-\frac{2}{3}}}$ $\frac{1}{x}=x^{-1}$

$$= 3\lim_{x\to\infty} x^{-1-(-\frac{2}{3})} = 3\lim_{x\to\infty} x^{-\frac{1}{3}}$$

$$= 3\lim_{x\to\infty} \frac{1}{\sqrt[3]{x}}$$

$$= \frac{3}{\lim_{x\to\infty} \sqrt[3]{x}} = \frac{3}{\infty} = 0 \quad \#$$

3: Indeterminate difference.

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$$

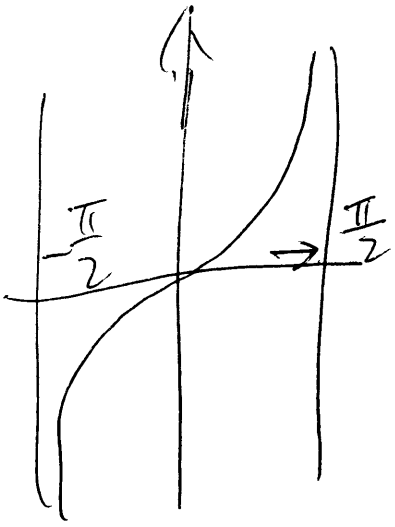
$$= \infty - \infty$$

where we suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$.

"$\infty - \infty$" may exist, or may not exist.

Ex 4: $\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x)$

Solution: $= \lim_{x \to \frac{\pi}{2}^-} \sec x - \lim_{x \to \frac{\pi}{2}^-} \tan x$



$= \lim_{x \to \frac{\pi}{2}^-} \frac{1}{\cos x} - \lim_{x \to \frac{\pi}{2}^-} \tan x$

$= \frac{1}{\lim_{x \to \frac{\pi}{2}^-} \cos x} - \infty$

$= \infty - \infty$ #

So it is indeterminate difference.

$$\lim_{x \to \frac{\pi}{2}^-} (\sec x - \tan x) = \lim_{x \to \frac{\pi}{2}^-} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x} \right)$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin x}{\cos x}$$

$$= \frac{\lim\limits_{x\to\frac{\pi}{2}^-}(1-\sin x)}{\lim\limits_{x\to\frac{\pi}{2}^-}\cos x}$$

$$= \frac{1-\lim\limits_{x\to\frac{\pi}{2}^-}\sin x}{\lim\limits_{x\to\frac{\pi}{2}^-}\cos x}$$

$$= \frac{1-1}{0} = \frac{0}{0}.$$

it is indeterminate!

then: $\lim\limits_{x\to\frac{\pi}{2}^-}\frac{1-\sin x}{\cos x} = \lim\limits_{x\to\frac{\pi}{2}^-}\frac{(1-\sin x)'}{(\cos x)'}$

$$= \lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{-\sin x} = \lim_{x\to\frac{\pi}{2}^-}\frac{\cos x}{\sin x} = \frac{0}{1} = 0$$ #

To deal with the indeterminate difference, we try to convert the difference to a quotient by using a common denominator. If $f(x)$ and $g(x)$ are fractions.

or

4: Indeterminate powers.

$$\lim_{x \to a} [f(x)]^{g(x)}$$

① If $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$

then: the type of $[0^0]$ is indeterminate

② If $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$

then: the indeterminate type of $\boxed{\infty^0}$

③ If $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \pm\infty$,

then, the indeterminate type of $\boxed{1^\infty}$

Ex: $\lim_{x\to 0^+} (1+\sin(4x))^{\cot x}$.

Solution: Because $\lim_{x\to 0^+} (1+\sin(4x)) = 1+0 = 1$

$$\lim_{x\to 0^+} \cot x = \lim_{x\to 0^+} \frac{\cos x}{\sin x} = \frac{\lim_{x\to 0^+} \cos x}{\lim_{x\to 0^+} \sin x}$$

$$= \infty,$$

Thus, we get $\boxed{1^\infty}$: indeterminate power.

Why 0^0 is indeterminate?

Let $y = 0^0$

$\Downarrow$

$\ln y = \ln 0^0 = 0 \ln 0$

$= 0 \cdot \lim_{x \to 0^+} \ln x$

$= -\boxed{0 \cdot \infty}$

$\Downarrow$

indeterminate product.

Why ∞^0 is indeterminate?

Let $y = \infty^0$

$\Downarrow$

$\ln y = \ln \infty^0 = 0 \cdot \ln \infty$

$= 0 \cdot \lim_{x \to \infty} \ln x$

$= \boxed{0 \cdot \infty}$

$\Downarrow$

Indeterminate product.

why 1^{∞} is indeterminate?

Let $y = 1^{\infty}$

$\Downarrow$

$\ln y = \ln 1^{\infty} = \infty \cdot \ln 1$

$= \boxed{\infty \cdot 0}$

$\Downarrow$

indeterminate product

"0^0, ∞^0, 1^∞ : Indeterminant

what of 0^∞?

Let $Y = 0^\infty$

$\Downarrow$

$\ln Y = \ln 0^\infty = \infty \cdot \ln 0$

$= -\boxed{\infty \cdot \infty}$

$\Downarrow$

$\ln Y = -\infty$

$e \quad e^{\cdot} =$

$\Downarrow$

$Y = e^{-\infty} = \lim_{x \to -\infty} e^x = 0$

$$x \ln x = \frac{x}{\frac{1}{\ln x}} \longrightarrow \frac{0}{0}$$

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{x}{\frac{1}{\ln x}} = \lim_{x \to 0^+} \frac{x'}{\left(\frac{1}{\ln x}\right)'}$$

$$= \lim_{x \to 0^+} \frac{1}{\frac{1' \cdot \ln x - 1 \cdot (\ln x)'}{(\ln x)^2}} = \lim_{x \to 0^+} \frac{1}{\frac{-\frac{1}{x}}{(\ln x)^2}}$$

$$= -\lim_{x \to 0^+} x (\ln x)^2$$

Wrong answer because of wrong choice.

Use logarithmic ~~differentiation~~ properties

Let $y = (1+\sin(4x))^{\cot x}$

$\Rightarrow \ln y = \ln(1+\sin(4x))^{\cot x}$

$\ln y = \cot x \cdot \ln(1+\sin(4x))$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \left(\cot x \ln(1+\sin(4x))\right)$

$\ln\left(\lim_{x\to 0^+} y\right) = \lim_{x\to 0^+} \cot x \cdot \lim_{x\to 0^+} \ln(1+\sin(4x))$

$= \infty \cdot 0$

2nd determinate product

Indeterminate "product"

Change the product to be a quotient

In general, $f(x)\,g(x) = \dfrac{f(x)}{\frac{1}{g(x)}} = \dfrac{g(x)}{\frac{1}{f(x)}}$

Thus:

$$\ln\left(\lim_{x\to 0^+} y\right) = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\frac{1}{\cot x}}.$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x}. \quad \text{"}\frac{0}{0}\text{"}$$

$$\star = \lim_{x\to 0^+} \frac{(\ln(1+\sin 4x))'}{(\tan x)'}$$

$$= \lim_{x \to 0^+} \frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x}$$

$$= \frac{\frac{4\cos 0}{1+\sin 0}}{\left(\frac{1}{\cos 0}\right)^2}$$

$$= \frac{\frac{4 \cdot 1}{1+0}}{1^2} = 4$$

thus: $\ln\left(\lim_{x \to 0^+} y\right) = 4$

$$\Rightarrow \lim_{x \to 0^+} y = \boxed{e^4} \#$$

Ex6: $\lim_{x\to 0^+} x^x$ "0^0"

Let $y = x^x$

$\Rightarrow \ln y = \ln x^x = x \ln x$. $\to 0\cdot(-\infty) = -0\cdot\infty$

$\Rightarrow \ln y = \dfrac{\ln x}{\frac{1}{x}}$ $\to \dfrac{-\infty}{+\infty}$

$\Rightarrow \lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}}$ "$\dfrac{\infty}{\infty}$"

$\Rightarrow \ln\left(\lim_{x\to 0^+} y\right) = \lim_{x\to 0^+} \dfrac{(\ln x)'}{\left(\frac{1}{x}\right)'}$

$= \lim_{x\to 0^+} \dfrac{\frac{1}{x}}{-\frac{1}{x^2}} \cdot = \lim_{x\to 0^+} (-x) = 0$

$$\Rightarrow \lim_{x\to 0^+} y = e^0 = 1$$

$$\Rightarrow \lim_{x\to 0^+} x^x = 1 \quad \#.$$

Summary:

① Indeterminate quotient $\Leftarrow \frac{0}{0}$ or $\frac{\infty}{\infty}$

L'Hospital's rule

② Indeterminate product $\Leftarrow f(x)\,g(x) = \frac{f(x)}{\frac{1}{g(x)}} = \frac{g(x)}{\frac{1}{f(x)}}$ $\quad 0\cdot\infty$

③ Indeterminate difference $\Leftarrow \infty - \infty$

change to quotient

④ Indeterminate powers: $0^0, 1^\infty, \infty^0$.

Logorithmic pwp.