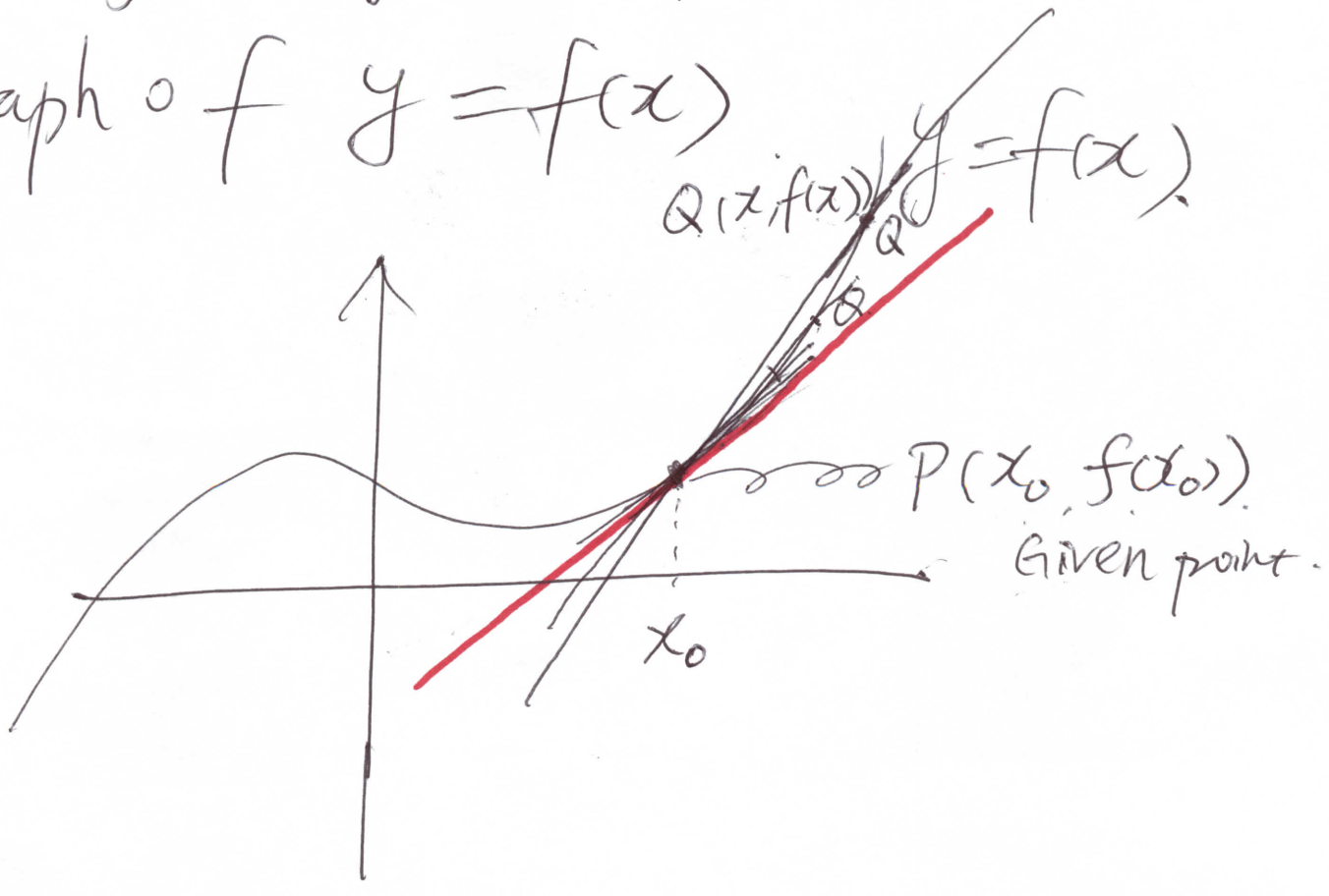
Ch2: Derivatives.

§2.1 The "tangent" problem

$\Longleftrightarrow$ find out the tangent line of a given function $y=f(x)$ through a given point on the graph of $y=f(x)$



Recall:

The equation of Tangent Line.

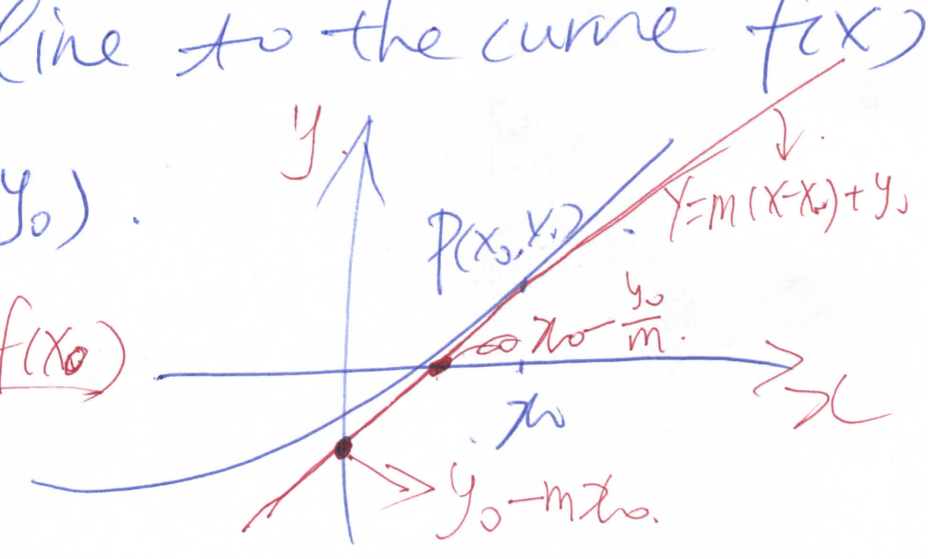
$$y = m(x - x_0) + f(x_0).$$

or: $y = m(x - x_0) + y_0$ → Given

where, the point $P(x_0, y_0)$ is on the curve of $y = f(x)$ (Given)

$m = f'(x_0)$ is the slope of the tangent line to the curve $f(x)$ through $P(x_0, y_0)$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



For a line equation:

like the one on the left:

$y = m(x - x_0) + y_0$.

① : x- intercept : at which $f(x) = 0$.

i.e., let $y = 0 \Rightarrow m(x - x_0) + y_0 = 0$

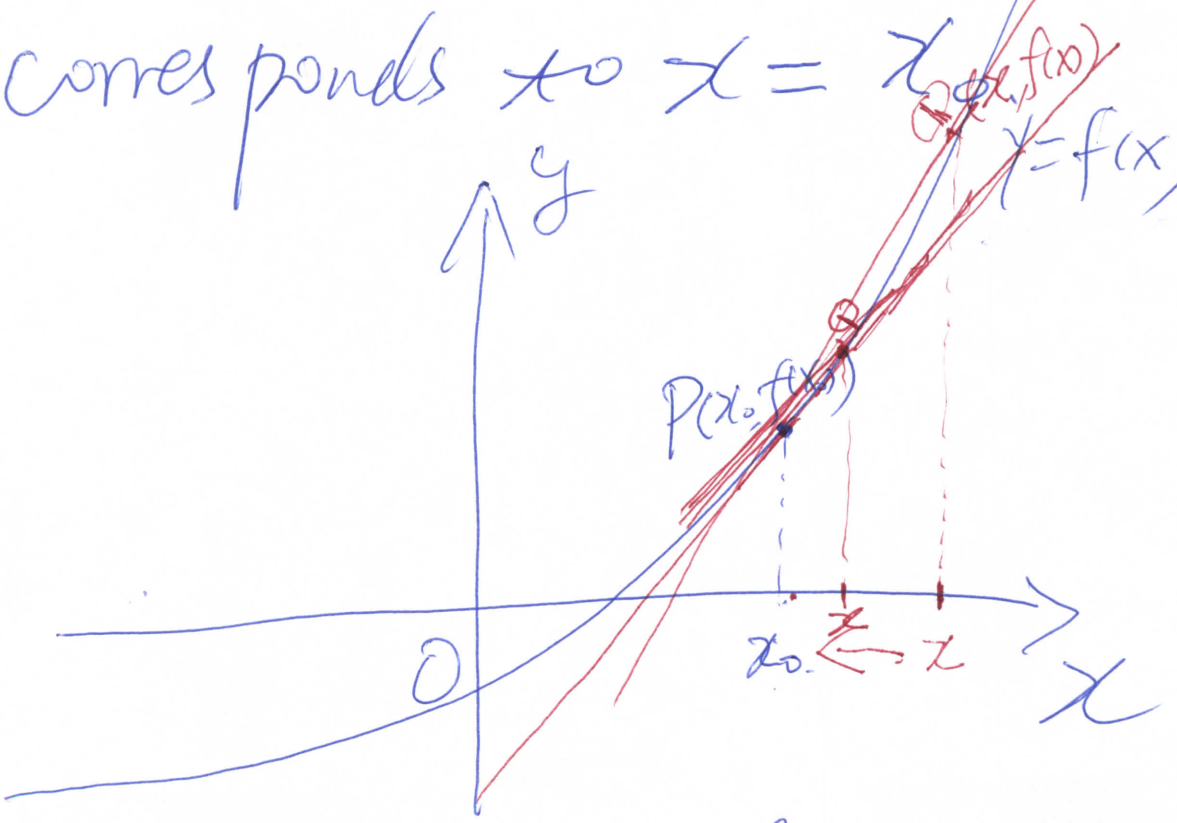
$\Rightarrow x = \frac{mx_0 - y_0}{m} = \boxed{x_0 - \frac{y_0}{m}}$.

② : y- intercept : by simply let $x = 0$;

$\Rightarrow y = m(0 - x_0) + y_0$

$= \boxed{y_0 - mx_0}$.

Given a function $f(x)$ and a point P on the curve of $y=f(x)$ which corresponds to $x=x_0$



Draw a tangent line to this curve through the point $P(x_0, \underbrace{f(x_0)}_{y_0})$

Find the equation of this tangent line!!!

$$y - y_0 = m(x - x_0)$$

$$\Downarrow$$

$$\boxed{m = \frac{y - y_0}{x - x_0}}$$

which implies we need another $Q(x, y)$.

"create"?

on the curve $y = f(x)$

to highest order term of x.

① Create a secant line $\overline{PQ}$ cutting the graph of $y=f(x)$ through $P(x_0, f(x_0))$ and $Q(x, f(x))$, for $x \neq x_0$.



Then, the slope of $\overline{PQ}$ is:

$$m_{\overline{PQ}} = \frac{f(x) - f(x_0)}{x - x_0}$$

② Move Q to P along the graph of $y=f(x)$ by moving $x \to x_0$

In general, the equation of a line is defined as:

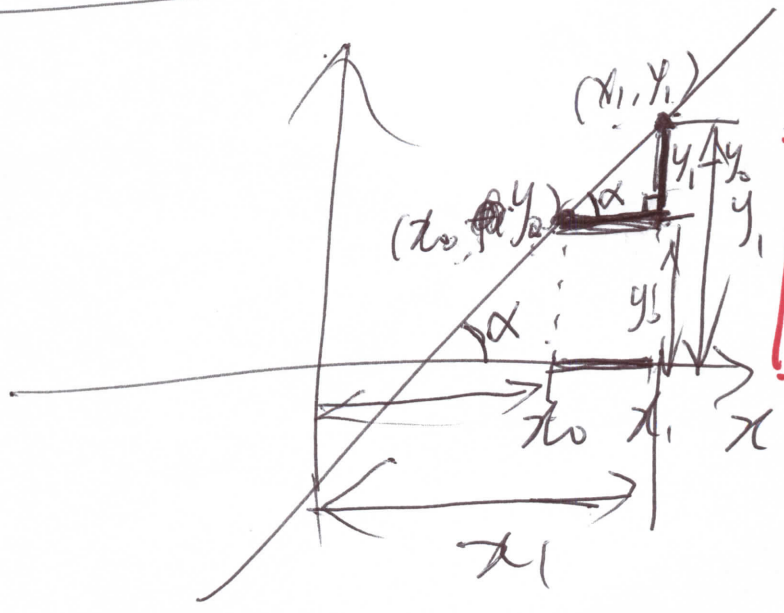
$$y_0 = f(x_0)$$

$$y = m(x - x_0) + y_0$$

Where, ① m : slope.

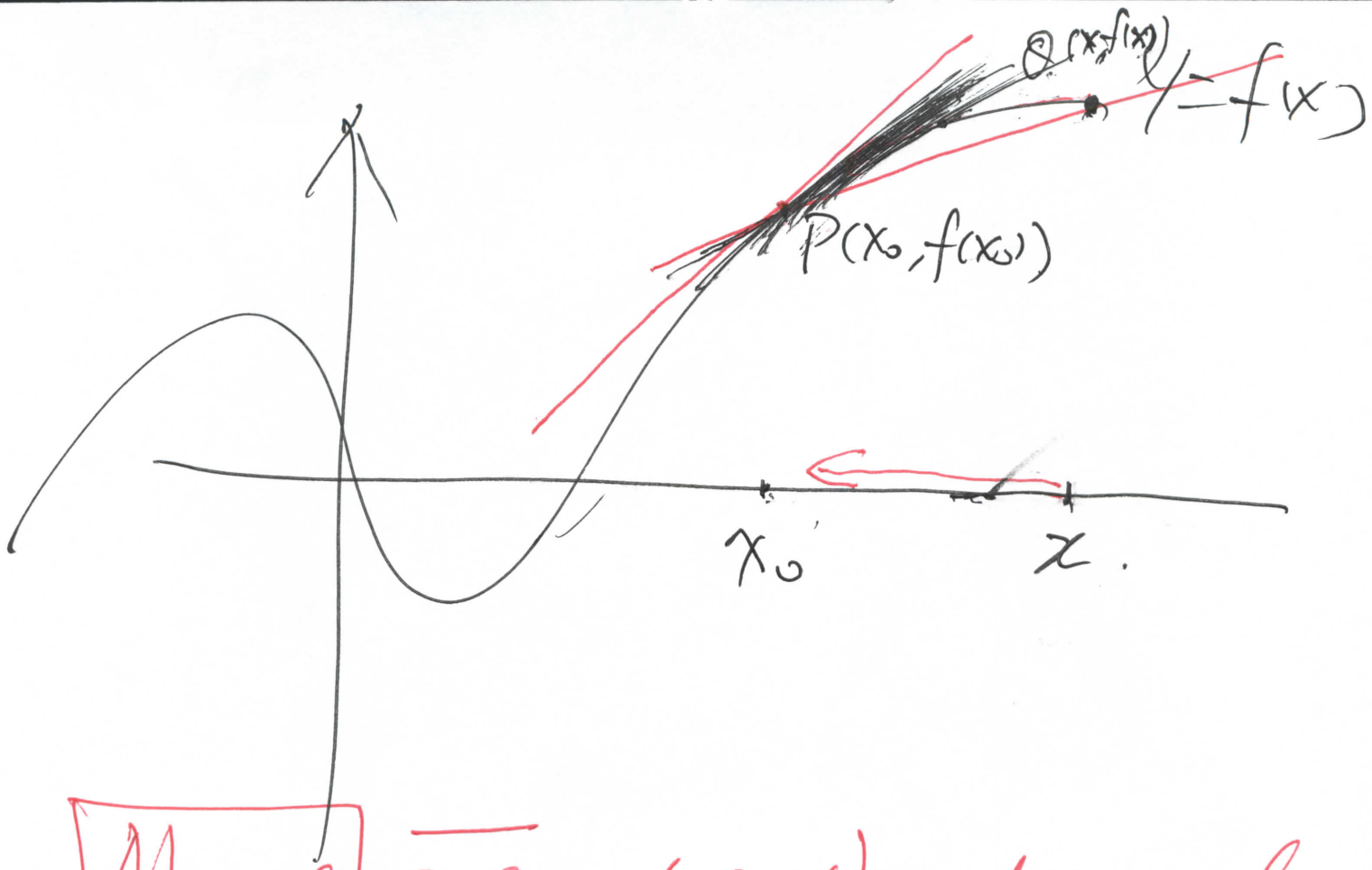
② (x_0, y_0) is a point on the line.

i.e., (x_0, y_0) satisfies the equation of line.



$$m = \tan\alpha$$

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$



Move $\overline{PQ}$ to the tangent line.

by moving $Q \to P$ if we

(on the graph of $y = f(x)$)

let $x \to x_0$.

$\lim \overline{PQ}$ = The tangent line

$Q \to P$

or $x \to x_0$

Then, the equation of tangent line is:

$$y = \left(\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}\right)(x-x_0) + f(x_0)$$

$$\Rightarrow y = f'(x_0)(x-x_0) + f(x_0)$$

where the slope of tangent

is just the derivative of $f(x)$

at "x_0" ; $f'(x_0)$.

Eventually. $\overline{PQ}$ becomes the tangent line. as $Q \to P$ or as $x \to x_0$.

The slope of tangent line, m_P,

$$m_P = \lim_{Q \to P} m_{\overline{PQ}} = \lim_{x \to x_0} m_{\overline{PQ}}$$

$$m_P = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Let $f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

which is called the derivative of $f(x)$ at $x = x_0$.

Let $h = x - x_0 \Rightarrow h \to 0$ as $x \to x_0$

$\Downarrow$

$x = x_0 + h$

Substitute $x_0 + h$ for x in the definition of derivative.

$$\Rightarrow f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

Ex1: Find the equation of tangent line to $y = \frac{3}{x}$ at the point $(3, 1)$.

Solution: The point $(3,1)$ is on the graph of $y=\frac{3}{x}$, because $(3,1)$ satisfies $y=\frac{3}{x}$. $\therefore \boxed{x_0=3}$

$$m=\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$$

$$=\lim_{x\to 3}\frac{\frac{3}{x}-1}{x-3}$$

$$=\lim_{x\to 3}\frac{\frac{3-x}{x}}{x-3}=\lim_{x\to 3}\frac{-(x-3)}{x(x-3)}$$

$$=-\lim_{x\to 3}\frac{1}{x}=-\frac{1}{3}.$$

$$\Rightarrow f'(3)=-\frac{1}{3}.$$

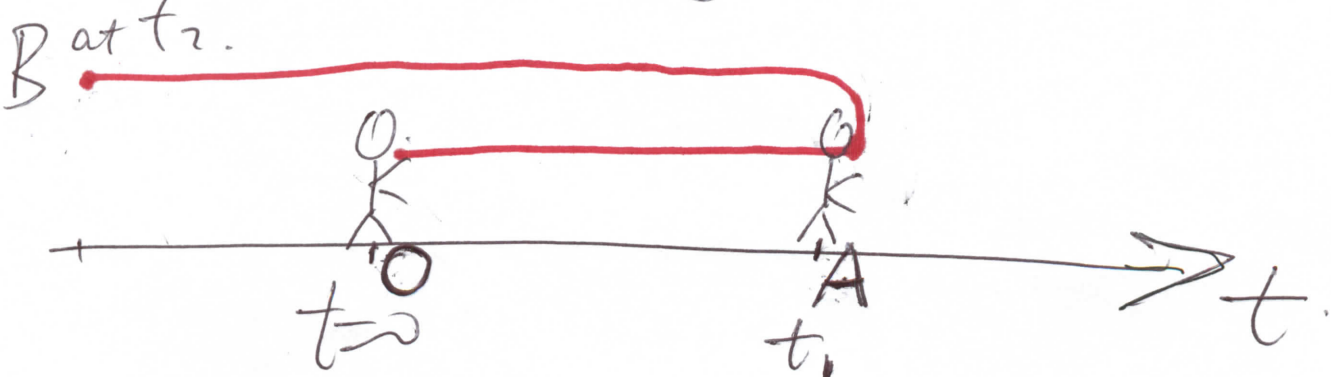
Then, the equation of tangent line:

$$y = -\frac{1}{3}(x-3) + 1$$

$$\Rightarrow y = -\frac{x}{3} + 1 + 1 = -\frac{x}{3} + 2$$

$$\Rightarrow x + 3y - 6 = 0 \quad \#$$

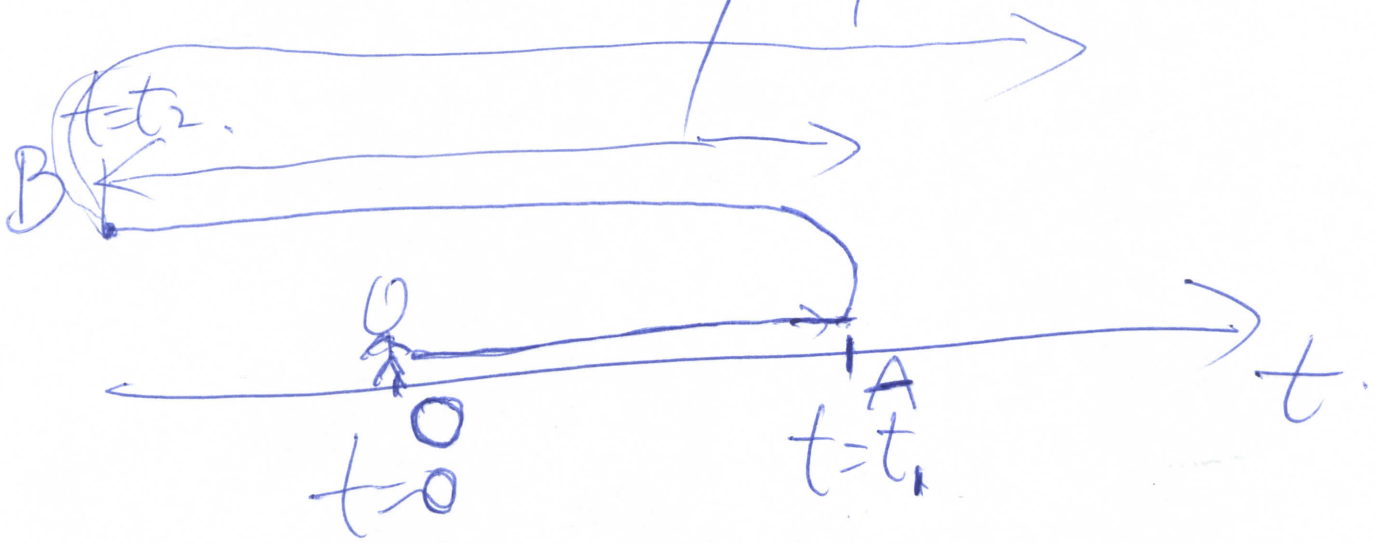
3. The velocity problem:



① Displacement: The terminal position subtracts the initial position. e.g. $A-O>0$, $B-O<0$

which can be positive can be negative.

3: the velocity problem



(initial position)

Positive direction: rightward

negative direction: leftward.

① : Displacement : the ~~terminal~~ current position subtracts the initial position.

position function:
notation: $S(t)$

~~another term~~: position function.

e.g.: $S(t_1) = A - O$ (positive)

$S(0)$

During $[0, t_1]$: the displacement // $S(t_2) = B - O$ (negative)

② Distance: The total distance.

"D" the particle travels during the released time period.

$D > 0$ ★

e.g.: During $[0, t_2]$,

$$D = |\overline{OA}| + |\overline{AB}|$$

③ ~~Velocity~~

Average Velocity: V:

$$V = \frac{S}{t} = \frac{\text{Displacement}}{\text{Time released}}$$

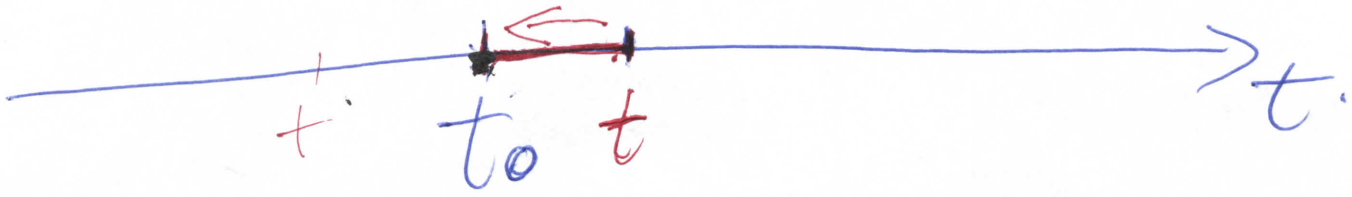
Recall: Speed $= \frac{\text{Distance}}{\text{Time released}} > 0$

Velocity: $V > 0$: moving forward (rightward)
(vector)

or $V < 0$: moving backward (leftward)

Remark: Speed = $\|\text{velocity}\|$.
(Scalar) "number" "$\|\cdot\| \Rightarrow$ Norm"

(4) Instantaneous velocity
at the time point "$t = t_0$"



$$V(t_0) = \lim_{t \to t_0} \frac{S(t) - S(t_0)}{t - t_0}$$

$$\boxed{V(t_0) = S'(t_0).} = \lim_{h \to 0} \frac{S(t_0+h) - S(t_0)}{h}$$
where $h = t - t_0$.

④ Acceleration, $a(t)$

$$a(t_0) = \lim_{t \to t_0} \frac{V(t) - V(t_0)}{t - t_0}$$

$$\boxed{\begin{aligned} a(t_0) &= V'(t_0) \\ V(t_0) &= S'(t_0) \end{aligned}}$$

where: $S(t)$ = displacement function

$V(t)$: Velocity function

$a(t)$: Acceleration function.

Ex1: Suppose a ball is dropped from the upper deck of CN Tower, 450m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution: (a) $v(5) = \lim_{h \to 0} \frac{S(5+h) - S(5)}{h}$

$= S'(5)$

where, $S(t) = \frac{1}{2} g t^2$, where $g = 9.8 m/s^2$

$= \frac{1}{2} \cdot 9.8 t^2 = 4.9 t^2$

$$V(5)=\lim_{h\to 0}\frac{4.9(5+h)^2-4.9\cdot 5^2}{h}$$

$$=\lim_{h\to 0}\frac{4.9\left[25+10h+h^2-25\right]}{h}$$

$$=4.9\lim_{h\to 0}\frac{10h+h^2}{h}$$

$$=4.9\lim_{h\to 0}(10+h)$$

$$=4.9\cdot(10+0)=49\,(m/s)$$

(b): Because $450=S(t)=4.9t^2$

$$\Rightarrow t^2=\frac{450}{4.9}\Rightarrow t=\sqrt{\frac{450}{4.9}}\approx 9.6$$

$$V(9.6)=4.9\lim_{h\to 0}\frac{(9.6+h)^2-9.6^2}{h}$$

$$= 4.9 \lim_{h \to 0} (19.2 + h).$$

$$= 4.9 \times 19.2$$

$$= 94 \text{ m/s.} \rightarrow \doteq 94 \text{ m/s.}$$

$$f'(a) = \lim_{h \to 0.} \frac{f(a+h) - f(a)}{h}$$

4: Derivative function.

Definition: $\boxed{f'(x)} = \lim_{h \to 0.} \frac{f(x+h) - f(x)}{h}$

where x is any number in the domain of $f(x)$.

5: the rate of change:

$\Delta x = x_2 - x_1$ ⇒ the change in x.

$\Delta y = y_2 - y_1$ ⇒ the change in y.

or: $\Delta y = f(x_2) - f(x_1)$

The difference quotient: $\frac{\Delta y}{\Delta x} = \frac{f(x_2)-f(x_1)}{x_2-x_1}$

⇒ called the average rate of change of $y = f(x)$ with respect to x over $[x_1, x_2]$.

⇒ the instantaneous rate of change

$$= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_1)$$

$x_2 \to x_1 \Leftrightarrow \lim_{h \to 0} \frac{f(x_1+h)-f(x_1)}{h} = f'(x_1)$

Derivative

The rate of change of position

with respect to time

$\Leftrightarrow$ velocity : $v(t) = s'(t)$

$V(t) = S'(t)$

Derivative

the rate of change of velocity

w.r.t. time

$\Leftrightarrow$ acceleration :

$a(t) = v'(t)$

$a(t) = V'(t)$

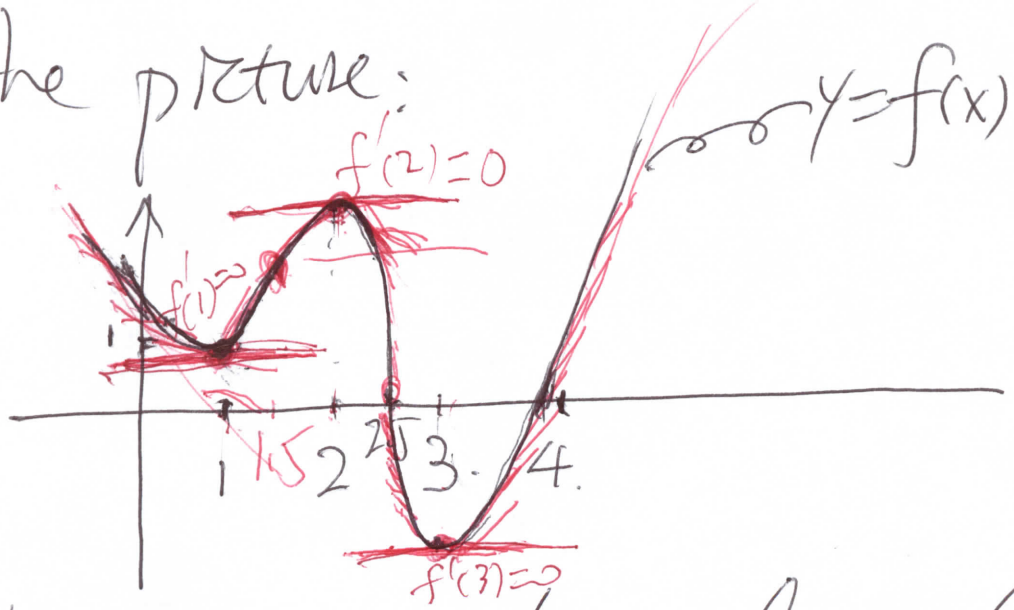
§2.2 The derivative as a function

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

* can be any number in the domain of $f(x)$.

⇒ called the derivative of $f(x)$.

Ex1: The function $f(x)$ is shown in the picture:



what does $f'(x)$ look like?

$f'(a)$: the slope of tangent line to the graph of $f(x)$ through $(a, f(a))$

Ex2: Given $f(x) = x^3 - x$.

find $f'(x)$.

Solution: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

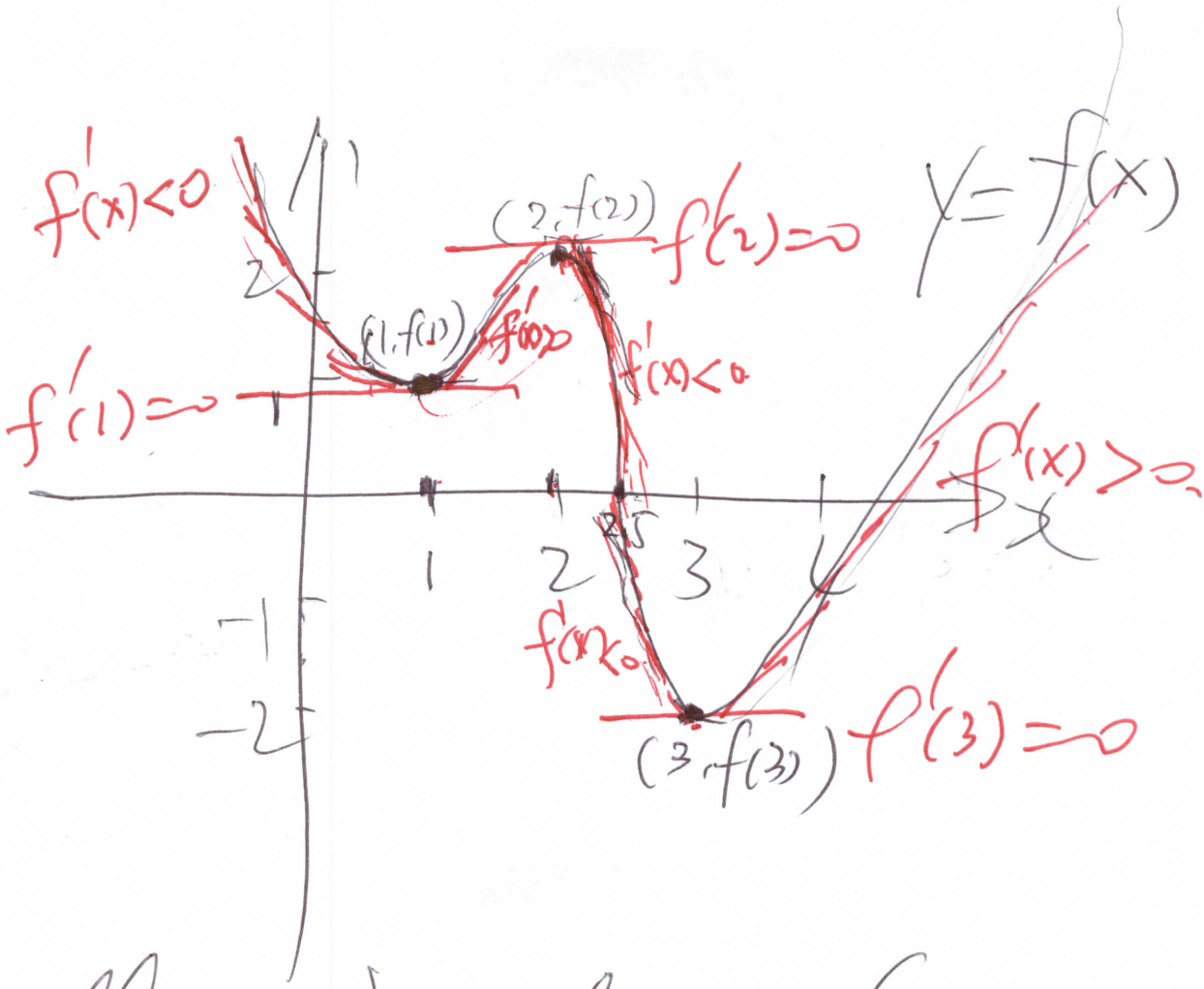
$= \lim_{h \to 0} \frac{(x+h)^3 - (x+h) - x^3 + x}{h}$

$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$

$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$

$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1)$.

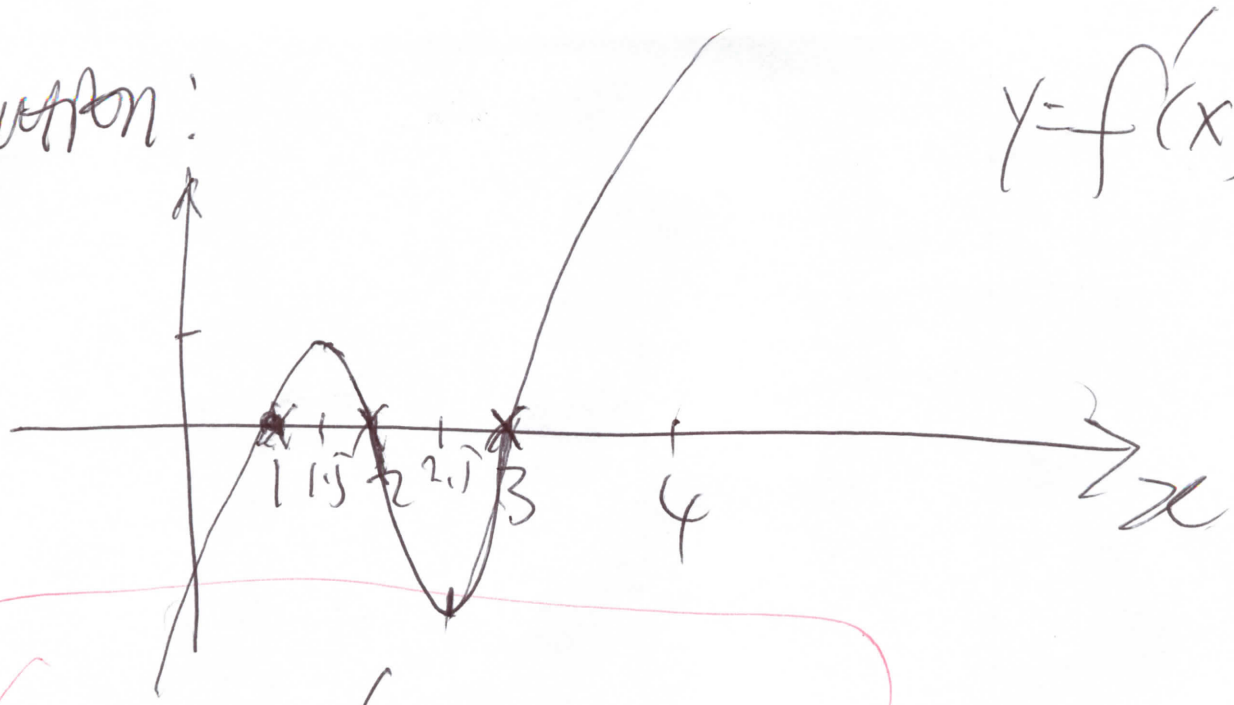
$= 3x^2 - 1$ #



Recall: the slope of a tangent line to the curve of $y=f(x)$ through a point $P(x_0, y_0)$.

is: $f'(x_0)$.

Solution:



$x=1$: $f'(1)=0$.

$x=2$: $f'(2)$ 不存在

$x=3$: $f'(3)=0$.

	$f(x)$	$f'(x)$
$x\le 1$	↘	↗
$1\le x\le 2$	↗	[1, 1.5] ↗
		[1.5, 2] ↘
$2\le x\le 3$	↘	[2, 2.5] ↘
		[2.5, 3] ↗
$x\ge 3$	↗	↗

Thus: $(x^3 - x)' = 3x^2 - 1$.

3: Other notations of derivative of $y = f(x)$

$f'(x) = y'$

"Leibniz" notation: $\boxed{\frac{dy}{dx}}$.

$$\boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

"Δ" $\to$ "d"

$$f'(a) = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a}$$

4: Differentiable functions.

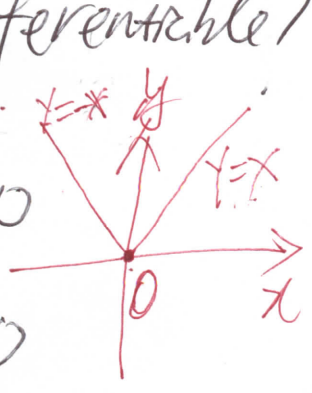
① A function $f(x)$ is differentiable at "a" if $f'(a)$ exists.

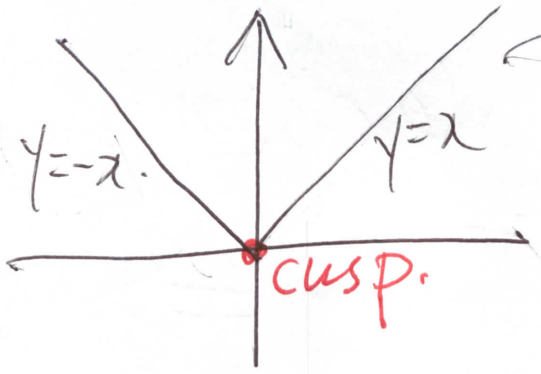
② $f(x)$ is differentiable on an

★ open interval (a,b) [or, (a,∞), "Domain" $(-\infty,a)$, $(-\infty,\infty)$] if it is differentiable at every number in this interval.

Ex3: Where is $f(x)=|x|$ differentiable?

Solution: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$





$\curvearrowright\curvearrowright\curvearrowright y=|x|.$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{|x+h|-|x|}{h}.$$

$$=\begin{cases}\lim_{h\to 0}\frac{\overbrace{x+h-x}^{=h}}{h}=1, & \forall\, x\geq 0\\ \lim_{h\to 0}\frac{\overbrace{-(x+h)-(-x)}^{=-h}}{h}=-1 & \forall\, x<0\end{cases}$$

$\Rightarrow f(x)$ is differentiable for $x>0$

and for $x<0$.

However: $\lim_{h\to 0^{+}}\frac{f(x+h)-f(x)}{h}$.

$$f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h}.$$

$$= \boxed{\lim_{h\to 0^{\pm}} \frac{|h|}{h}}.$$

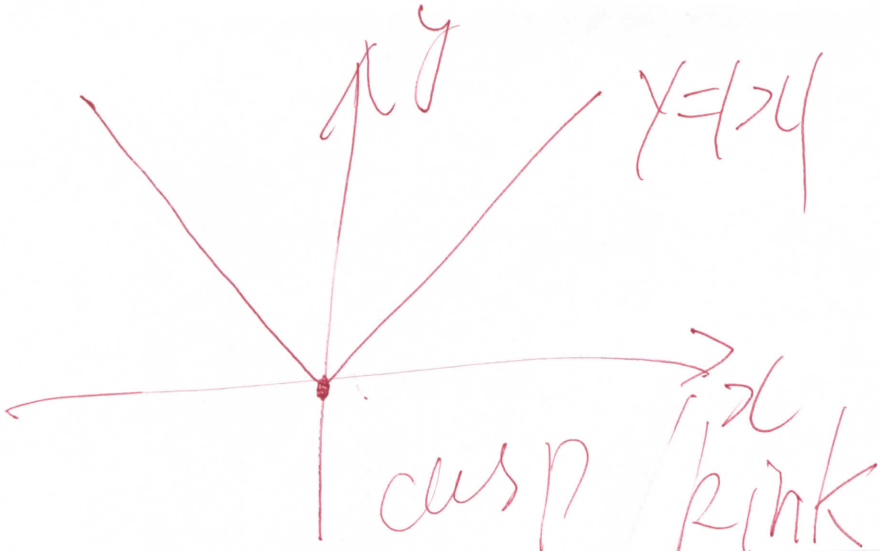
$$= \begin{cases} \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1 \\ \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1 \end{cases}$$

$$\Rightarrow \lim_{h\to 0} \frac{|h|}{h} \text{ does not exist}$$

$$\Rightarrow f'(0) \text{ does not exist}$$

$$\Rightarrow f(x) = |x| \text{ not differentiable at } x=0. \quad \#$$

$f(x) = |x|$ is differentiable on $(-\infty, 0) \cup (0, \infty)$.



Non-differentiable function

$\Rightarrow$ non-smooth.

"smooth": differentiable.

Logical thinking:

Given a statement:

If A, then B.

Then: ~~what is~~ which one is the following statement true?

① If B then A ✗

② If Non A then Non B ✗

③ If Non B then Non A. ✓

Theorem of Differentiable function.

(a) If $f(x)$ is differentiable

then $f(x)$ is continuous.

(b) If $f(x)$ is discontinuous

then $f(x)$ is not differentiable.

$f(x)$ is not differentiable at "a"

Remarks: ① $\Rightarrow$ $f(a)$ does not exist

$\Rightarrow$ $f(x)$ does not have a tangent line at $(a, f(a))$. ~~a~~ Non-smooth curve

$\Rightarrow$ The graph of $f(x)$ has "corner" or "kink" at the point $(a, f(a))$

① ~~If B then A~~

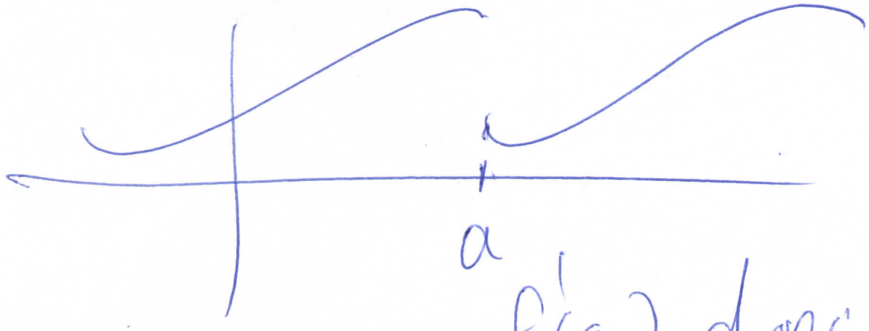
② ~~If Non A then Non B~~

③ If Non B then Non A. ✓



$y = |x|$

Continuous But Non-diff.



$f'(a)$ does not exist

5: Theorem:

If $f(x)$ is differentiable at "a", then $f(x)$ is continuous at "a".

A

B

Remarks: ① However, continuous func. may not be differentiable

② If is continuous function at "a" must be not differentiable at "a"

Differentiable function

⇓

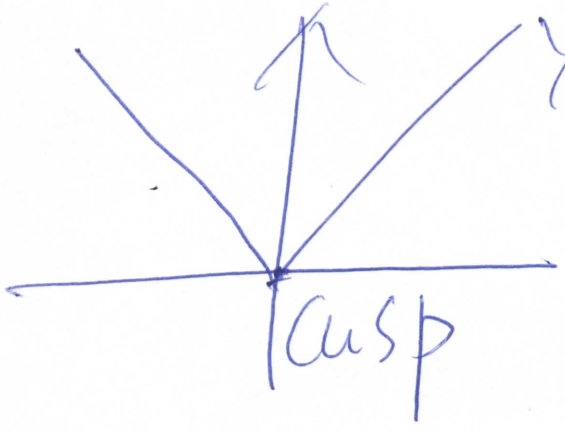
Smooth function.

or.

Non-differentiable function

⇓

Non-smooth function



$y = |x|$.

$$f'(x) = \frac{df}{dx} = \frac{dy}{dx}$$

The first derivative

first-order

A function.

$f'(x)$ is a function

and, if $f'(x)$ is differentiable

then it can be

differentiated again.

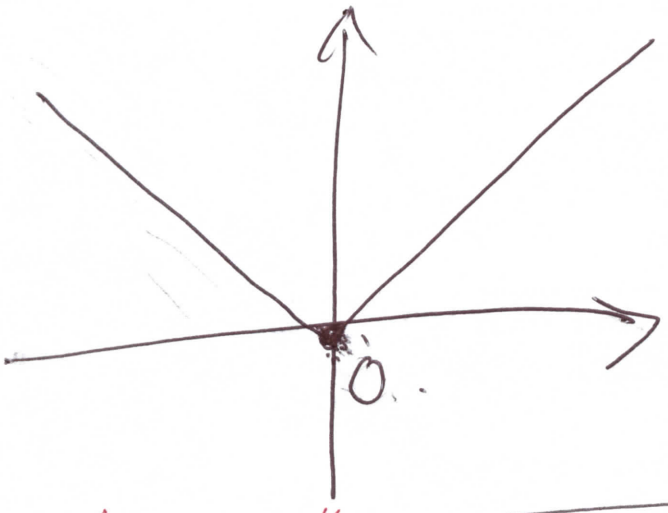
$$(f'(x))' = f''(x) = \frac{d^2y}{dx^2}.$$

$\Rightarrow$ Non-smooth function at $(a, f(a))$.

$\Rightarrow$ Not differentiable at $(a, f(a))$

i.e. Differentiable function $\Rightarrow$ Smooth function

e.g. $y = |x|$



"higher order"

Higher derivatives

$f'(x) = \frac{df}{dx}$: The first derivative.

$(f'(x))' = f''(x)$: The second derivative.

or: $\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} = f''(x) = y''(x)$

And: $f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$

$(f'(x))'$

The third derivative, $f'''(x)$, is defined as:

$$f'''(x) = (f''(x))' = \lim_{h\to 0} \frac{f''(x+h) - f''(x)}{h}$$

or: $f'''(x) = \frac{d^3y}{dx^3} = \frac{d^3f}{dx^3}$.

The fourth (order) derivative,

$$f^{(4)}(x) = (f'''(x))' = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h}.$$

or: $f^{(4)}(x) = \frac{d^4f}{dx^4} = \frac{d^4y}{dx^4}$

the n-th derivative.

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \frac{d^n y}{dx^n}.$$

And: $f^{(n)}(x) = \lim_{h \to 0} \frac{f^{(n+1)}(x+h) - f^{(n-1)}(x)}{h}$.

↳ the slope of the tangent line to the graph of $f^{(n-1)}(x)$ at a certain number "a" = $f^{(n)}(a)$. through the point $(a, f^{(n-1)}(a))$.

Ex : $f(x) = x^3 - 2x^2 + 3x + 4$.

find $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$.

Solution:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

$$= \lim_{h \to 0} \frac{(x+h)^3 - 2(x+h)^2 + 3(x+h) + 4 - (x^3 - 2x^2 + 3x + 4)}{h}.$$

($x^2 + 2xh + h^2$)

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2x^2 - 4xh - 2h^2 + 3x + 3h + 4 - x^3 + 2x^2 - 3x - 4}{h}.$$

$$= \lim_{h \to 0} (3x^2 + 3xh + h^2 - 4x - 2h + 3)$$

$$= 3x^2 - 4x + 3.$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{3(x+h)^2 - 4(x+h) + 3 - (3x^2 - 4x + 3)}{h}$$

$x^2 + 2xh + h^2$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 4x - 4h + 3 - 3x^2 + 4x - 3}{h}$$

$$= \lim_{h\to 0} \frac{6xh + 3h^2 - 4h}{h}$$

$$= \lim_{h\to 0} (6x + 3h - 4)$$

$$f''(x) = 6x - 4.$$

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}$$

$$= \lim_{h\to 0} \frac{6(x+h) - 4 - (6x-4)}{h}$$

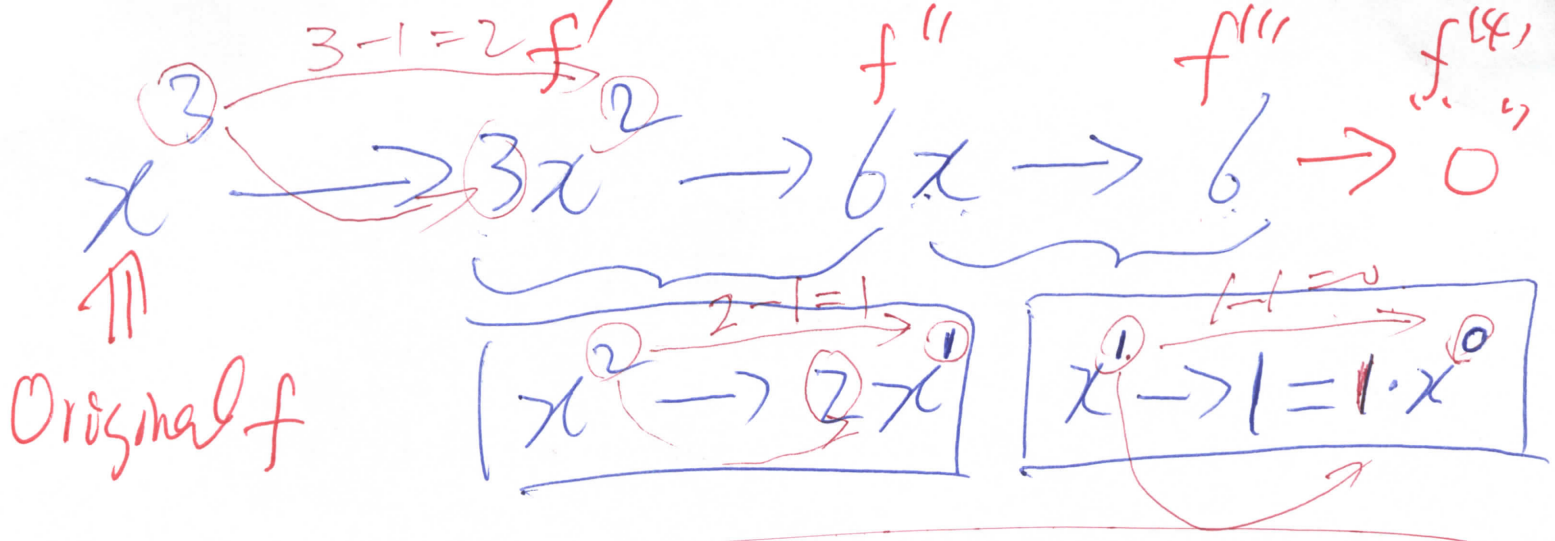
$$= \lim_{h\to 0} \frac{6x+6h-4-6x+4}{h}$$

$$= \lim_{h\to 0} 6$$

$$f'''(x) = 6.$$

$$f^{(4)}(x) = \lim_{h\to 0} \frac{f'''(x+h) - f'''(x)}{h}$$

$$= \lim_{h\to 0} \frac{6-6}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$



$$(x^n)' = n x^{n-1}$$

which is true for any real number "n".

the "power-rule"

$(x^3)' \Rightarrow 3x^2$ ✓

$(x^2)' \Rightarrow 2x$ ✓

$(x)' \Rightarrow 1$ ✓

$(C)' \Rightarrow 0$ ✓

CONSTANT

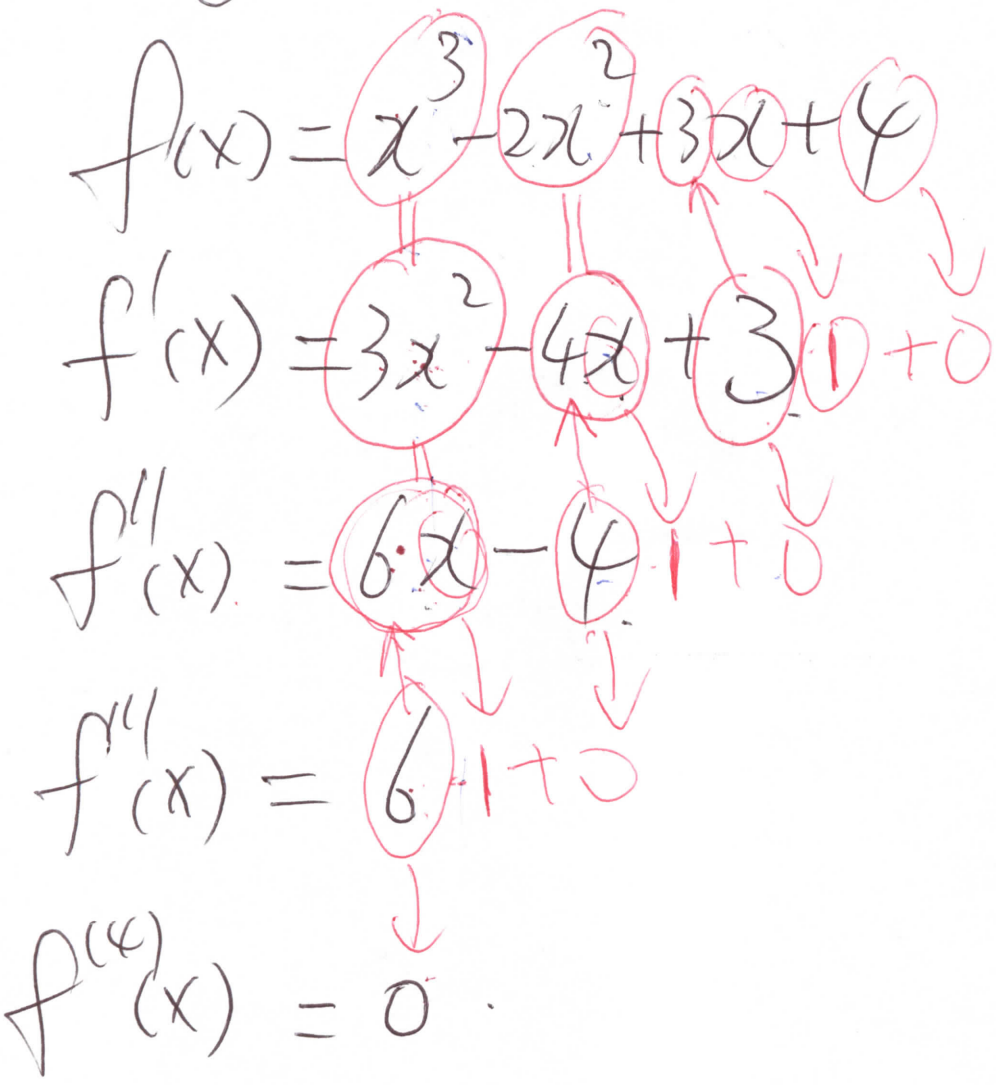
They can be verified:

e.g. $(x^2)' = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim_{h \to 0} (2x + h) = 2x$

Summary:



$f(x) = x^3 - 2x^2 + 3x + 4$

$f'(x) = 3x^2 - 4x + 3 + 0$

$f''(x) = 6x - 4 + 0$

$f'''(x) = 6 + 0$

$f^{(4)}(x) = 0$

$9 \quad 0 \quad 0$

$(x^3)' \to 3x^2$

$x^n \to n x^{n-1}$

$(3x^2)' \to 3 \; 2x^1 = 6x.$

$(6x^1)' \to 6 \cdot 1 x^0 = 6.$

$(-2x^2)' \to -2 \; 2x^1 = -4x$

$(-4x^1)' \to -4 \; 1x^0 = -4$

§2.3: Basic differentiation formulas

① $\frac{d}{dx}(c) = 0 \iff c' = 0$

② $\frac{d}{dx}(x) = 1 \iff (x)' = 1 = x^0$

③ $\frac{d}{dx}(x^2) = 2x \iff (x^2)' = 2x^1$

④ $\frac{d}{dx}(x^3) = 3x^2 \iff (x^3)' = 3x^2$

⑤ $\frac{d}{dx}(x^4) = 4x^3 \iff (x^4)' = 4x^3$

⑥ $\boxed{\frac{d}{dx}(x^n) = nx^{n-1} \iff (x^n)' = nx^{n-1}}$

$$(x^n)' = n x^{n-1}$$ ⇒ Power Rule

Where, n is any real number

~~but not "1" "11"~~

~~because: $x^{-1} = \frac{1}{x}$~~

~~⇒ (~~

~~Ex~~: Properties of Differentiation

(1) $\frac{d}{dx}(c f(x)) = c \frac{d}{dx} f(x)$

or: $(c f(x))' = c f'(x)$

e.g.: $(x^{\frac{1}{2}})' = (\sqrt{x})'$

$= \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}}$

$= \frac{1}{2} \frac{1}{\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}} = (\sqrt{x})'}$

$(x^{-\frac{1}{2}})' = -\frac{1}{2} x^{-\frac{1}{2}-1}$

$= -\frac{1}{2} x^{-\frac{3}{2}}$

$= -\frac{1}{2} \frac{1}{x^{\frac{3}{2}}}$

$= -\frac{1}{2} \frac{1}{\sqrt{x^3}} = -\frac{1}{2} \frac{1}{|x| \sqrt{x}}$

② $(f+g)' = f' + g'$

③ $(f-g)' = f' - g'$

Ex1: Given $y = x^4 - 6x^2 + 4$

find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}(x^4 - 6x^2 + 4)$

$= \frac{d}{dx}(x^4) + \frac{d}{dx}(-6x^2) + \frac{d}{dx}(4)$

$= 4x^3 + (-6)2x + 0$

$= 4x^3 - 12x$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(4x^3 - 12x)$$

$$= \frac{d}{dx}(4x^3) - \frac{d}{dx}(12x)$$

$$= 4\frac{d}{dx}(x^3) - 12\frac{dx}{dx}\frac{d}{dx}(x)$$

$$= 4 \cdot 3x^2 - 12 \cdot 1$$

$$= 12x^2 - 12.$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(12x^2 - 12)$$

$$= 12 \cdot 2x - (12)'$$

$$= 24x + 0 = 24x.$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \frac{d}{dx}(24x) = 24\frac{dx}{dx}$$

$$= 24 =$$

$$\frac{d^5y}{dx^5} = (24)' = 0$$ #

Given the n-th degree polynomial, its $(n+1)$-th derivative must be "zero". because its n-th derivative is constant.

Ex2: Given $f(x) = \frac{1}{x^2}$.

find $f'(x)$

Solution: $f(x) = \frac{1}{x^2} = x^{-2}$

$$\Rightarrow f'(x) = (x^{-2})'$$

$$= (-2)x^{-2-1}$$

$$= -2x^{-3} = -\frac{2}{x^3}.$$

$$f(x) = \sqrt[3]{x^2} = x^{\frac{2}{3}}$$

$$\Rightarrow f'(x) = \left(x^{\frac{2}{3}}\right)'$$

$$= \frac{2}{3} x^{\frac{2}{3}-1}$$

$$= \frac{2}{3} x^{-\frac{1}{3}}$$

$$= \frac{2}{3} \frac{1}{x^{\frac{1}{3}}}$$

$$= \frac{2}{3} \frac{1}{\sqrt[3]{x}}$$

$$\left(x^{\frac{2}{3}}\right)' = \frac{2}{3\sqrt[3]{x}}$$

$$\frac{d}{dx}(\sin x) = (\sin x)' = \cos x$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= 0$$

$$= \cos x$$

$$\frac{d}{dx}(\cos x) = \boxed{(\cos x)' = -\sin x}$$

Ex 4: Given $y = 3\sin\theta + 4\cos\theta$.

find $\frac{dy}{d\theta}$

Solution: $\frac{dy}{d\theta} = (3\sin\theta + 4\cos\theta)'$

$= (3\sin\theta)' + (4\cos\theta)'$

$= 3(\sin\theta)' + 4(\cos\theta)'$

$= 3\cos\theta - 4\sin\theta$

#.

Applications to Rate of Change

Ex: The position function of a particle is given as:

$$S(t) = t^3 - 6t^2 + 9t$$

(a) Find the velocity at time "t":

$$V(t) = S'(t) = (t^3 - 6t^2 + 9t)'$$

$$= (t^3)' - 6(t^2)' + 9t'$$

$$= 3t^2 - 6 \cdot 2t + 9 \cdot 1$$

$$= 3t^2 - 12t + 9. \#$$

$(t^n)' = nt^{n-1}$ for any "n" real number "n".

(b) What is the velocity after 2s?

$$V(2) = 3 \cdot 2^2 - 12 \cdot 2 + 9 = -3$$

After 4s?

$$V(4) = 3 \cdot 4^2 - 12 \cdot 4 + 9 = 9$$

(c) When is the particle at rest?

$$V(t) = 0$$

$$\Rightarrow 3t^2 - 12t + 9 = 0.$$

$$\Rightarrow 3(t^2 - 4t + 3) = 0$$

$$\Rightarrow t^2 - 4t + 3 = 0$$

$$\Rightarrow (t-1)(t-3) = 0.$$

$$\Rightarrow t = 1 \text{ or } t = 3.$$

After 1s and 3s the particle is at rest.

(d): When is the particle moving forward?

$$V(t) > 0$$

$$3t^2 - 12t + 9 > 0.$$

$$\Downarrow$$

$$3(t-1)(t-3) > 0$$

$$\Downarrow$$

$$(t-1)(t-3) > 0.$$

$$\Downarrow$$

$$\begin{cases} t-1 > 0 \\ t-3 > 0 \end{cases} \quad \text{or} \quad \begin{cases} t-1 < 0 \\ t-3 < 0 \end{cases}$$

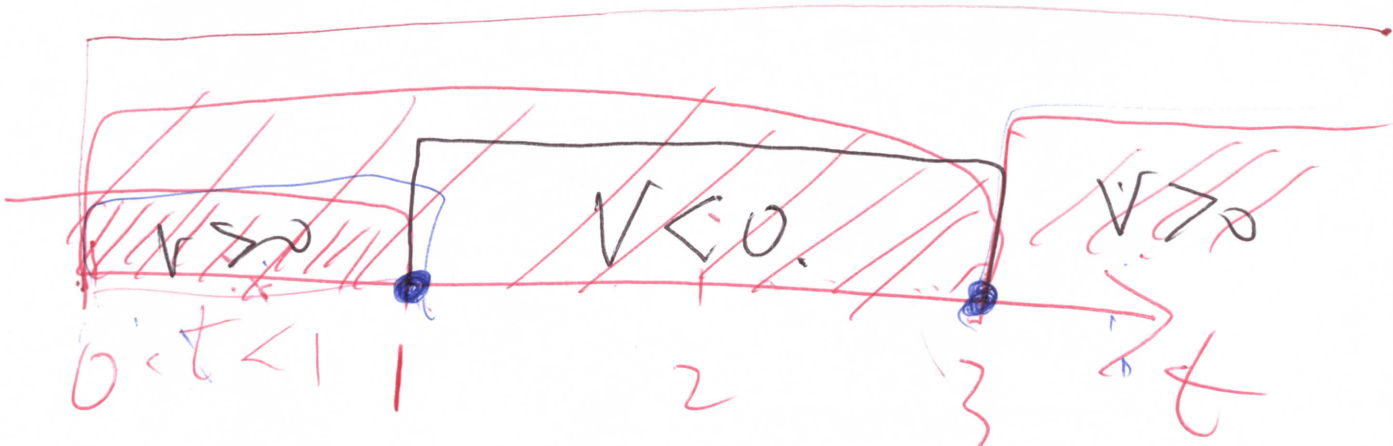
$$\Downarrow \qquad\qquad \Downarrow$$

$$\begin{cases} t > 1 \Rightarrow (1, \infty) \\ t > 3 \Rightarrow (3, \infty) \end{cases} \qquad \begin{cases} t < 1 \Rightarrow [0, 1) \\ t < 3 \Rightarrow [0, 3) \end{cases}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$t > 3 \Rightarrow (3, \infty). \qquad t < 1 \Rightarrow [0, 1)$$

when $t<1$ or $t>3$ the velocity is positive $\Rightarrow$ moving forward.



intersection of two intervals.

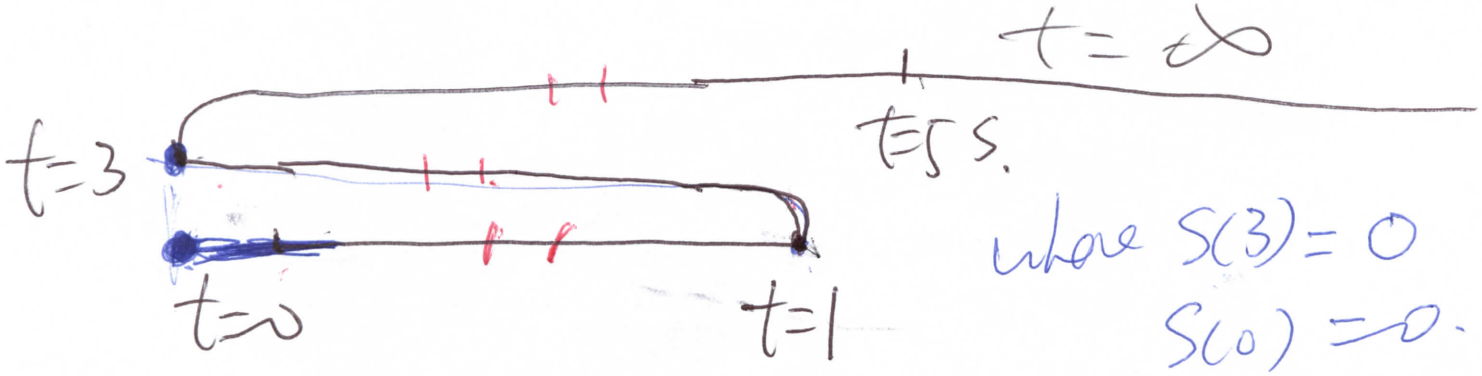
(e) When is the particle moving backward? $\Leftrightarrow V(t)<0$.

$\Rightarrow \boxed{1<t<3}$

"intersection"

$(t-1)(t-3)<0 \Rightarrow \begin{cases} t>1 \\ t<3 \end{cases}$ or $\begin{cases} t<1 \\ t>3 \end{cases}$

(f) Draw a diagram to represent the motion of the particle.

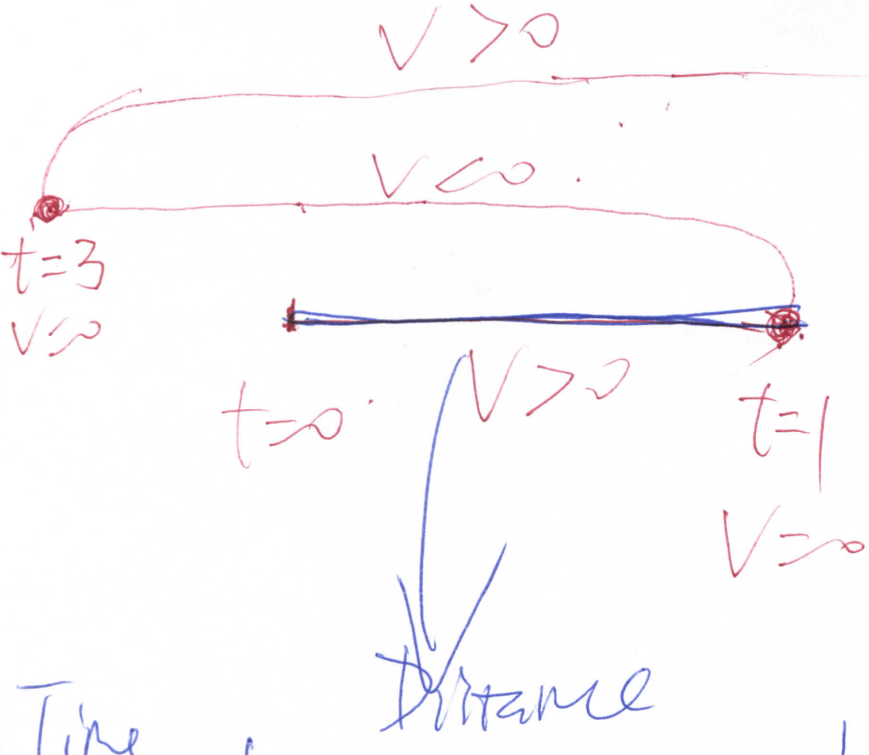


where $S(3) = 0$
$S(0) = 0$.

(g) Find the total distance traveled by the particle during the first 5 seconds.

① $|S(1) - S(0)|$: the distance between $t=0$ and $t=1$

$= |(1-6+9) - 0| = 4$



Time ⟶ Distance

$[0,1]: |S(1) - S(0)|$

$+$

$(1,3]: |S(3) - S(1)|$

$+$

$[3,5]: |S(5) - S(3)|$

$\|$

Total distance.

② $|S(3) - S(1)|$: the distance in $[1, 3]$

$$\| \quad \text{in } [1, 3]$$

$$|3^3 - 6\cdot 3^2 + 9\cdot 3 - 4| = |-4| = 4$$

$3^3 - 6\cdot 3^2 + 9\cdot 3 = 0$

③ $|S(5) - S(3)|$: the distance in $[3, 5]$

$$\|$$

$$|5^3 - 6\cdot 5^2 + 9\cdot 5 - 0| = |20 - 0| = 20$$

Thus, the total distance $= 4 + 4 + 20$

$= 28$.

§2.4: The product and Quotient Rule of Differentiation

1: The product rule:

$$\frac{d}{dx}(f(x)\cdot g(x)) = (fg)' = f'g + fg'$$

proof: skipped.

$$(fg)' = \lim_{h\to 0} \frac{\cancel{(fg)(x+h)} \quad f(x+h)g(x+h) - f(x)g(x)}{h}$$

Ex1: Differentiate

$$f(t) = \sqrt{t}(3+2t)$$

$$\Rightarrow f'(t) = (\sqrt{t})'(3+2t) + \sqrt{t}(3+2t)'$$

$(t^{\frac{1}{2}})' = \frac{1}{2}t^{-\frac{1}{2}}$

$$= \frac{1}{2\sqrt{t}}(3+2t) + \sqrt{t}\cdot 2$$

$$= \frac{3+2t}{2\sqrt{t}} + 2\sqrt{t}$$

$$= \frac{3+2t+4t}{2\sqrt{t}}$$

$$= \frac{(3+6t)\sqrt{t}}{2\sqrt{t}\sqrt{t}} = \frac{3\sqrt{t}+6t\sqrt{t}}{2t}$$

Ex2: If $h(x) = x\,g(x)$ and

$g(3) = 5$, $g'(3) = 2$.

find $h'(3)$

Solution: $h'(x) = (x\,g(x))'$

$= x'g(x) + x\,g'(x)$

$= g(x) + x\,g'(x)$

$\Rightarrow$ let $x = 3$:

$h'(3) = g(3) + 3 \cdot g'(3)$

$= 5 + 3 \cdot 2$

$= 5 + 6 = 11$ #

2: The quotient rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$$

$$= \frac{f'g - fg'}{g^2(x)}$$

or: $$\boxed{\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$$ ☆

Ex3: Differentiate $y = \frac{\sqrt{x}}{1+x^2}$.

$$\Rightarrow y' = \frac{dy}{dx} = \left(\frac{\sqrt{x}}{1+x^2}\right)' = \frac{(\sqrt{x})'(1+x^2) - \sqrt{x}(1+x^2)'}{(1+x^2)^2}$$

$$= \frac{\frac{1}{2\sqrt{x}}(1+x^2) - \sqrt{x}\cdot 2x}{(1+x^2)^2} \cdot \frac{(2\sqrt{x})}{(2\sqrt{x})} = \frac{1+x^2-4x^2}{2\sqrt{x}(1+x^2)^2}$$

$$= \frac{\frac{1-3x^2}{2\sqrt{x}}}{(1+x^2)^2}$$

$$= \frac{1-3x^2}{2\sqrt{x}(1+x^2)^2} \quad \#$$

3: Derivatives of Trigonometry.

① $(\sin x)' = \cos x$.

② $(\cos x)' = -\sin x$.

③ $\boxed{(\tan x)' = \sec^2 x.}$

because $\tan x = \frac{\sin x}{\cos x}$

$\cos x$ $\quad$ $-\sin x$

$$\Rightarrow (\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \left(\frac{1}{\cos x}\right)^2 = \sec^2 x.$$

④ $(\cot x)' = -\csc^2 x.$

⑤ $\boxed{(\sec x)' = \tan x \sec x}$

Because $\sec x = \frac{1}{\cos x}$

$$\Rightarrow (\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{1'\cos x - 1\cdot(\cos x)'}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x}$$

$$= \boxed{\tan x \sec x}$$

⑥ $(\csc x)' = -\cot x \csc x$

Ex4: Differentiate $f(x) = \dfrac{\sec x}{1+\tan x}$

$$\Rightarrow f'(x) = \frac{(\sec x)'(1+\tan x) - \sec x (1+\tan x)'}{(1+\tan x)^2}$$

$$= \frac{\tan x \sec x (1+\tan x) - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

$1+\tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$$= \frac{\tan x \sec x + \tan^2 x \sec x - \sec^3 x}{(1+\tan x)^2}$$

$(\sec^2 x - 1)\sec x$

$$= \frac{\tan x \sec x + \sec^3 x - \sec x - \sec^3 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2} \quad \#$$

$$\tan^2 x = \sec^2 x - 1$$

$$1 + \tan^2 x = \sec^2 x$$

Exs: For what values of x does the graph of $f(x) = \frac{\sec x}{1+\tan x}$ have a horizontal tangent?

Solution: $f'(x) = 0 \Rightarrow$ zero slope $\Rightarrow$ horizontal tangent line

$$\frac{\sec x(\tan x + 1)}{(1+\tan x)^2} = 0$$

$$\sec x(\tan x + 1) = 0$$

$\Rightarrow \sec x = \infty$ or $\tan x = 1$

$\Downarrow$ $\Downarrow$

$\frac{1}{\cos x} = 0$ $\quad x = \frac{\pi}{4} + n\pi$

$\Downarrow$

Not possible

☆ (n is any integer)

§2.5 The chain Rule

there is for function like:

$(f \circ g)(x)$

$(f \circ g \circ h)(x)$

Composite function

§2.5 Chain Rule

Apply to the function:

Composition function, e.g.

$$(f \circ g)(x) = f(g(x)).$$

$$\Downarrow$$

the chain rule:

$$\frac{d}{dx}(f(g(x)) = f'(g(x)) \cdot g'(x).$$

or: $(f(g(x))' = f'(g(x)) \cdot g'(x)$

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

$$\Downarrow$$

$$(f(g(h(x))))' = f'(g(h(x)) \cdot g'(h(x)) \cdot h'(x)$$

$$(f \circ g \circ h \circ p)(x) = f(g(h(p(x))))$$

$$\Downarrow$$

$$\frac{d}{dx}(f \circ g \circ h \circ p)(x) = \frac{d}{dx}(f(g(h(p(x)))))$$

$$= f'(g(h(p(x)))) \cdot g'(h(p(x)) \cdot h'(p(x)) \cdot p'(x)$$

Ex1: Differentiate $y = \sin(x^2)$.

$$\Rightarrow \frac{dy}{dx} = (\sin(x^2))'$$

$$= \sin'(x^2) \cdot (x^2)'$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x\cos(x^2) \ \#$$

Ex2: Given $y = \sin^2(x) \Rightarrow y = g(\sin x)$

Now $g = x^2$.

$= (\sin x)^2$

$$\Rightarrow \frac{dy}{dx} = ((\sin^2(x))'$$

where, $y = f(g(x))$

and $g(x) = \sin x$

$f(x) = x^2$.

Let $y = \sin^2 x = u^2$

and $u = \sin x$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{dy}{dx}$$

$$= \frac{d(u^2)}{du} \cdot \frac{d(\sin x)}{dx}$$

$$= 2u \cdot \cos x \cdot$$

$$= 2 \sin x \cos x \cdot$$

$$= 2\sin x \cdot (\sin x)'$$

$$= 2\sin x \cdot \cos x.$$

$$= \sin 2x \quad \#$$

Or: Let $y = u^2$, $u = \sin x$.

$$y = (\sin x)^2 = \sin^2(x)$$

Then: $\frac{dy}{dx} = \frac{du^2}{du} \cdot \frac{d\sin x}{dx}$.

$$= 2u \cdot \cos x$$

$$= 2\sin x \cos x$$

$$= \sin 2x.$$

Ex3: Given $g(t)=\left(\frac{t-2}{2t+1}\right)^9$

Find $g'(t)$

Solution: Let $g(t)=u^9$, $u=\frac{t-2}{2t+1}$

$$\Rightarrow g'(t)=\frac{du^9}{du}\cdot\frac{du}{dt}$$

$$=9u^8\cdot\frac{d}{dt}\left(\frac{t-2}{2t+1}\right)$$

$$=9\cdot\left(\frac{t-2}{2t+1}\right)^8\cdot\frac{(t-2)'(2t+1)-(t-2)(2t+1)'}{(2t+1)^2}$$

$$=9\cdot\left(\frac{t-2}{2t+1}\right)^8\cdot\frac{2t+1-2t+4}{(2t+1)^2}$$

$$=9\cdot\left(\frac{t-2}{2t+1}\right)^8\cdot\frac{5}{(2t+1)^2}$$

$$=\frac{45(t-2)^8}{(2t+1)^{10}}\quad\#.$$

Ex4: Given $y=\sqrt{\sec(x^3)}$

find y'

Solution: let $y=\sqrt{u}$, $u=\sec v$

$v=x^3$.

$$\Rightarrow y' = \frac{d}{du}\sqrt{u}\cdot\frac{d\sec v}{dv}\cdot\frac{dx^3}{dx}$$

$$= \frac{1}{2\sqrt{u}}\cdot\tan v\sec v\cdot 3x^2$$

$$\star = \frac{1}{2\sqrt{\sec(x^3)}}\cdot\tan(x^3)\sec(x^3)\cdot 3x^2$$

$$= \frac{3x^2\tan(x^3)\sec(x^3)}{2\sqrt{\sec(x^3)}} \quad \#.$$

$$= \frac{3}{2} x^2 \tan(x^3) \left(\sec(x^3)\right)^{\frac{+1}{2}} \quad \#$$

Ex5: If $f(x) = \sin(\cos(\tan(\sec x)))$

find $\frac{df}{dx}$.

Solution: $f'(x) = \sin'(\cos(\tan(\sec x))) \cdot$

$\cos'(\tan(\sec x)) \cdot$

$\tan'(\sec x) \cdot \sec'(x)$

$= -\cos(\cos(\tan(\sec x))) \cdot$

$(+\sin)(\tan(\sec x)) \cdot$

$\sec^2(\sec x) \cdot \tan x \sec x.$

$\#$

~~$u = \cos(\tan(\sec x))$~~

$f(x) = \sin u$.

$u = \cos v$

$v = \tan w$

$w = \sec x$

Then $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}$

$= \cos u \cdot (-\sin v) \cdot \sec^2 w \cdot \tan x \sec x$

$= \cos(\cos(\tan(\sec x))) \cdot (-\sin(\tan(\sec x))) \cdot \sec^2(\sec x) \cdot \tan x \cdot \sec x$

§2.6 Implicit Differentiation.

Function: ① Explicit function.

② Implicit function.

⇓

Differentiate such function ⇒

① Any function which can be written as $y = f(x)$ is called explicit function about x.

② Implicit functions: always given as an equation

for examples:

$$x^2+y^2=25=5^2$$

which is a circle at center(0,0) with the radius "5".

To write this function as $y=f(x)$, we have to solve $x^2+y^2=25$ for "y".

$$y^2=25-x^2 \Rightarrow y=\pm\sqrt{25-x^2}$$

How to find $\frac{dy}{dx}$ for

that implicit function

which is impossible to find

its explicit form ?

Ex1: Given $x^2+y^2=25$.

And $\frac{dy}{dx}$.

Solution: Step 1: Differentiate both sides of the equation

w.r.t. "x".

"with respect to"

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25) = 0$$

$$\Downarrow$$

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$$

y as dependent variable: $y = y(x)$

thus: $y^2 = y^2(x)$

$$\Downarrow$$

A composite function with $y = u^2$, $u = y(x)$

$$\Downarrow$$

$y^2(x)$.

Chain Rule $\Downarrow$ $\Downarrow$

$$2x + \boxed{2y \cdot \frac{dy}{dx}} = 0$$

Solve this equation for $\frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\Downarrow$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad \#$$

$$\frac{dy}{dx} = \frac{du^2}{du} \cdot \frac{dy(x)}{dx}$$

$$= 2u \cdot \frac{dy}{dx}$$

$$= 2y \cdot \frac{dy}{dx}$$

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} = 0$$

$$\Downarrow \qquad \Downarrow$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

"y" : dependent variable

$$y = y(x)$$

thus $y^2 = y^2(x)$: composite fun.

$$\frac{d(y^2)}{dx} = \frac{d}{dx}(y^2(x))$$

$$= 2\,y(x) \cdot y'(x)$$

Ex2: Find y'' for $x^4+y^4=16$.

Solution: ① Find $y'=\frac{dy}{dx}$ first:

$$\frac{d}{dx}(x^4+y^4)=\frac{d}{dx}(16)$$

$$\Downarrow$$

$$\frac{d(x^4)}{dx}+\frac{d(y^4)}{dx}=0$$

$$\Downarrow$$

$$4x^3+4y^3\frac{dy}{dx}=0 \quad \text{or} \quad \frac{d}{dx}(y^4(x))$$

$$\Downarrow$$

$$\frac{dy}{dx}=-\frac{4x^3}{4y^3}=-\frac{x^3}{y^3}.$$

$$y'' = \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)' = \left(-\frac{x^3}{y^3}\right)'$$

$$= -\frac{(x^3)' \cdot y^3 - x^3 \cdot (y^3)'}{y^6} \cdot \frac{dy}{dx}$$

$$= -\frac{3x^2 y^3 - x^3 \cdot 3y^2 \cdot \frac{dy}{dx}}{y^6}$$

$$= -\frac{3x^2 y^3 - x^3 \cdot 3y^2 \cdot \left(-\frac{x^3}{y^3}\right)}{y^6 \to y^4}$$

$$= -\frac{3x^2 y + 3x^3 \cdot \frac{x^3}{y^3}}{y^4}$$

$$= -\frac{\frac{3x^2 y^4 + 3x^6}{y^3}}{y^4} = -\frac{3x^2(y^4 + x^6)}{y^7}$$

$$= -\frac{3x^2 \cdot 16}{y^7} = -48\frac{x^2}{y^7} \quad \#$$

e.g., $\sin(y^2) + y^4 + 3y^3 + 2x^2 + 2\sin x + x = 5$

$\not\equiv$

$\boxed{y = f(x)}$ ← explicit function

implicit function as long as it is not given as $y = f(x)$ in the first place.

Find $\frac{dy}{dx}$

$$\frac{d}{dx}(\sin(y^2(x))) + \frac{d}{dx}y^4 + \frac{d}{dx}(3y^3) + \frac{d}{dx}(2x^3) + \frac{d}{dx}(2\sin x) + \frac{d}{dx}x = \frac{d}{dx}5 = 0.$$

$$\cos(y^2)\cdot 2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} + 9y^2\frac{dy}{dx} + 4x + 2\cos x + 1 = 0.$$

$$\frac{dy}{dx}(2y\cos(y^2) + 4y^3 + 9y^2) = -4x - 2\cos x - 1$$

$$\frac{dy}{dx} = \frac{-4x - 2\cos x - 1}{2y\cos(y^2) + 4y^3 + 9y^2} \quad \#$$

§2.7: Related Rates (of change)

At least, you are given more than one rate of change

derivative w.r.t. "time"

Ex1: A water tank has the shape of an inverted circular cone with base radius 2m and height 4m. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is 3m deep

Strategy of solving "Related Rates"

1: Read the problem carefully

2: Draw a diagram if possible.

3: Introduce notations, assign symbols to all quantities that are functions of time.

4: Express the given information and the unknown information in terms of derivatives.

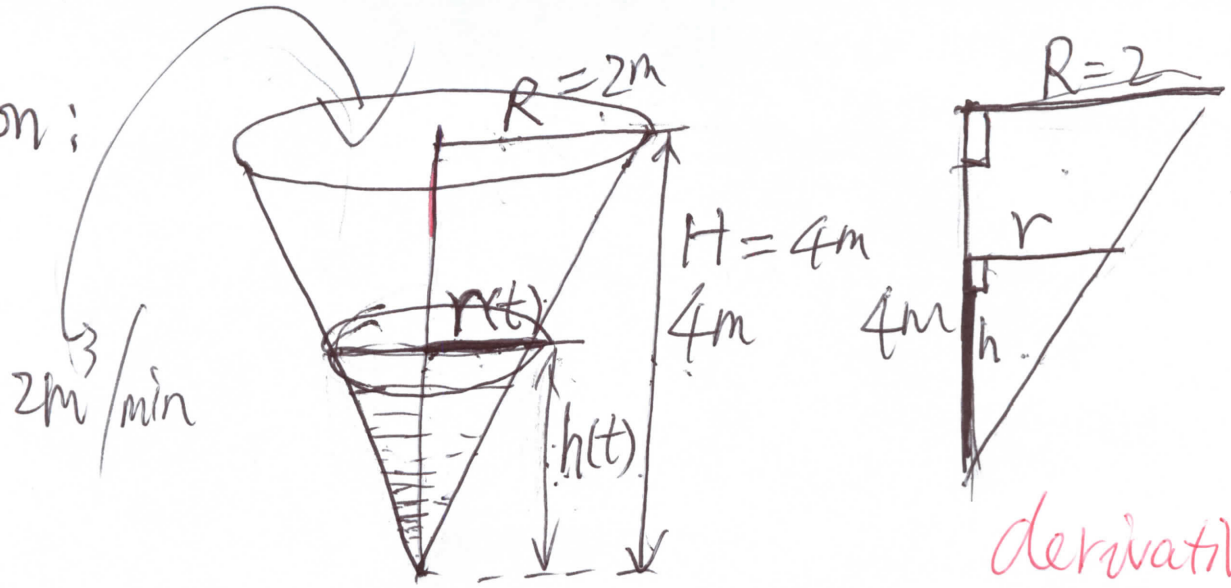
5: Write an equation that relates the ~~variables~~ various quantities of the problem (if necessary,

use the geometry of the situation. to eliminate one of the variables by substitution).

6: Use the Chain Rule to differentiate both sides of the equation w.r.t. "t".

7: Substitute the given info into the resulting equation and solve for the unknown rate.

Solution:



Given rate: $2\,m^3/min$. : the rate of change of volume.

derivative

m^3 → volume of water that fits the shape of cone.

$$\Rightarrow V = \frac{1}{3}Sh = \frac{1}{3}\pi r^2 h.$$

Further: $V(t) = \frac{1}{3}\pi r^2(t) h(t)$

$$2\,m^3/min = \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi r^2(t) h(t)\right)$$

Unknown rate: the rate of change of the height $h(t)$

$$\Rightarrow \boxed{\frac{dh}{dt}} = ?$$

These two rates, $\frac{dV}{dt}$ and $\frac{dh}{dt}$

give you two relates quantities.

$\boxed{V}$ and $\boxed{h}$

To eliminate r, we need to find out the relationship between r and h.

Similar triangles..

$$\Rightarrow \frac{R}{r} = \frac{H}{h} \Rightarrow \frac{2}{r} = \frac{4}{h}$$

$$\Rightarrow 4r = 2h \Rightarrow \boxed{r = \frac{1}{2}h}$$

Thus:

$$V(t) = \frac{\pi}{3}\left(\frac{1}{2}h(t)\right)^2 h(t)$$

$$= \frac{\pi}{3}\,\frac{1}{4}\,h^2(t)\,h(t)$$

$$\Rightarrow V(t) = \frac{\pi}{12}h^3(t)$$

$$\Rightarrow \frac{d}{dt}V(t) = \frac{\pi}{12}\frac{d}{dt}h^3(t)$$

$$2 =$$

$$\Rightarrow 2 = \frac{dV}{dt} = \frac{\pi}{12} 3h^2 \boxed{\frac{dh}{dt}} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{2}{\frac{\pi}{4} h^2}$$

$$= \frac{2 \cdot 4}{\pi} \cdot \frac{1}{h^2}$$

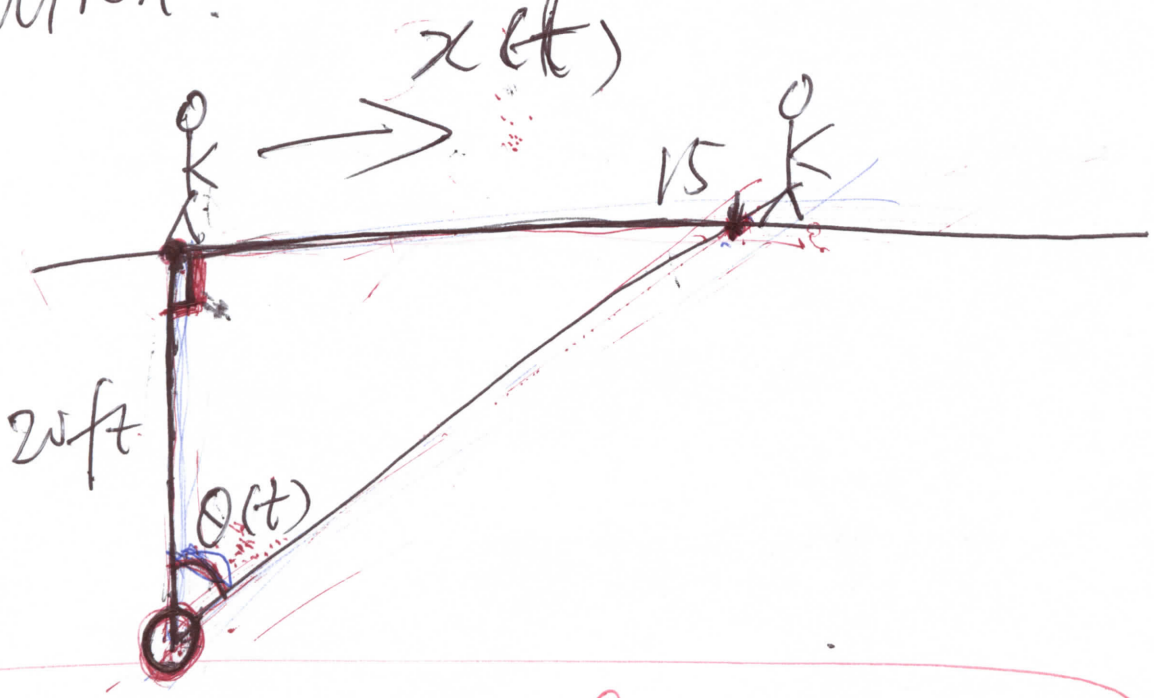
Since we are looking for $\frac{dh}{dt}$

at $h = 3$

$$\Rightarrow \frac{dh}{dt} = \frac{2 \cdot 4}{\pi} \cdot \frac{1}{3^2} = \frac{8}{9\pi} \quad \#$$

Ex 2: A man walks along a straight path at a speed of 4 ft/s. A search light is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the search light rotating when the man is 15 ft from the point on the path closest to the ~~second~~ search light?

Solution:



Given rate : $\frac{dx}{dt} = 4$ ft/s.

Unknown rate : $\frac{d\theta}{dt} = ?$

$$\Rightarrow \tan\theta = \frac{x}{20}.$$

$$\Rightarrow \frac{d}{dt}(\tan(\theta(t))) = \frac{d}{dt}\left(\frac{x(t)}{20}\right)$$

$$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

$$\Rightarrow \sec^2\theta \frac{d\theta}{dt} = \frac{1}{20} \cdot 4 = \frac{1}{5}.$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{1}{\sec^2\theta}.$$

Became $\boxed{\sec^2\theta = 1 + \tan^2\theta}$

and $\tan\theta = \frac{15}{20}.$ $\Rightarrow$

$$\Rightarrow \sec^2\theta = 1 + \left(\frac{15}{20}\right)^2$$

$$= 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16}.$$

$$= \frac{25}{16}$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{1}{\frac{25}{16}} = \frac{1}{5} \cdot \frac{16}{25} = \boxed{\frac{16}{125}} \#$$

$= 0.128$ (rad/s)

§2.8: Linear Approximation and Differentials.

Approximation: inaccurate calculation ☆ but with small error and that error is bounded. under control won't be concerned.

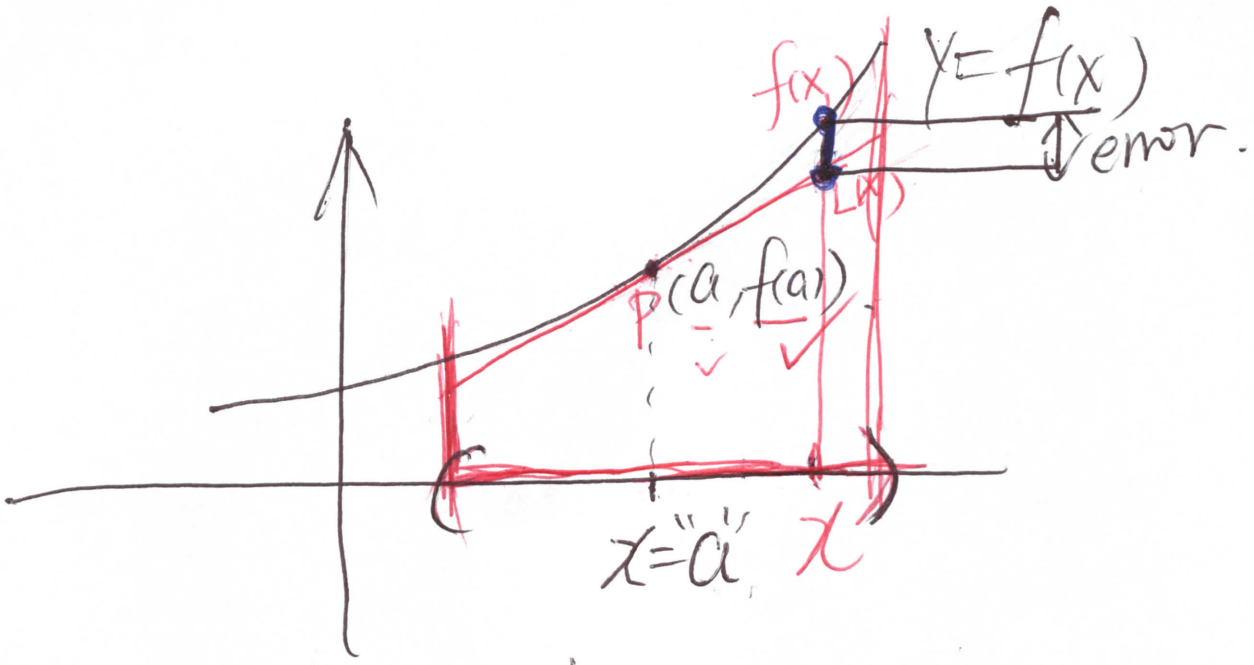
when the $y=f(x)$ is difficult to be calculated, we then consider such linear approximation.

e.g. $y = \sin(\ln(x))$

Given $x =$ "1.15" or $-$"0.11"

How to find their function values: $y(1.15)$, or $y(-0.11)$

Difficult without using calculator.



To find "$f(x)$", approximately near V: "$x=a$",

for a given value of $x=$"a",

and $f(a)$ can be easily calculated.

We draw the tangent line to the graph of $y=f(x)$ through the point $(a, f(a))$, whose equation is:

$$y = f(a) + f'(a)(x-a) = L(x)$$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a).$$

when x approaches to "a", the approximation becomes more accurate.

when $x = a$, the approximation becomes exact $\Leftrightarrow f(a) = L(a)$.

when x is away from "a", the approximation becomes ~~[illegible]~~ less accurate.

We call such approximate as

linear approximation:

$$f(x) \approx f(a) + f'(a)(x-a).$$

which is also called the

linearization of $f(x)$.

Ex1: Find the linearization of

$f(x) = \sqrt{x+3}$ at $a = 1$.

And use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

Solution: $a = 1 \Rightarrow f(a) = \sqrt{1+3} = \sqrt{4}$

$= 2$

$\Rightarrow f(a) = f(1) = 2$

$\Rightarrow f'(x) = \left(\sqrt{x+3}\right)' = \frac{1}{2\sqrt{x+3}} \cdot (x+3)'$

$= \frac{1}{2\sqrt{x+3}}.$

$\Rightarrow f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{2 \cdot 2} = \frac{1}{4}.$

Thus:

$$f(x) \approx f(1) + f'(1)(x-1)$$

$$\Rightarrow f(x) \approx 2 + \frac{1}{4}(x-1)$$

$$\Rightarrow \boxed{f(x) \approx \frac{x}{4} + \frac{7}{4}}$$

$\sqrt{3.98} = \sqrt{0.98+3}$ comparing with

$f(x) = \sqrt{x+3} \Rightarrow x = 0.98$

Thus: $f(0.98) \approx \frac{0.98}{4} + \frac{7}{4}$

$\sqrt{3.98} =$

$= 0.245 + 1.75$

$= 1.995$ #

Thus $\sqrt{3.98} \approx 1.995$

$\sqrt{4.05} = \sqrt{1.05+3} \Rightarrow x = 1.05$

To find $f(1.05) = \sqrt{4.05}$, we use its linearization to approximately calculate:

$$\sqrt{4.05} \approx \frac{1.05}{4} + \frac{7}{4} = \frac{8.05}{4}$$

$$= 2.0125 \quad \#$$

Differential

Given $y = f(x)$ where $f(x)$ is differentiable then:

① dx is called the differential of "x" for the independent variable x.

② dy is called the differential of the dependent variable "y".

Recall the derivative notation:

$$\frac{dy}{dx} = f'(x)$$

$$\Downarrow$$

$$dy = f'(x)\,dx$$

In particular, when $x = a$, then

$$dy\Big|_{x=a} = f'(a)\,dx \text{ at } x = a$$

where, $dx = \Delta x$, the difference in "x" change.

Because: $dx = \Delta x = x - a$.

$$\Downarrow$$

$$x = dx + a$$

$$\Downarrow$$

$$f(x) = f(dx + a).$$

The linear approximation:

$$f(x) \approx f(a) + f'(a)\underbrace{(x-a)}_{dx}$$

$dy = f'(a)dx$

can be rewritten as:

$$f(x) = f(a + dx) \approx f(a) + dy.$$

$$\Downarrow$$

$$f(x) \approx f(a) + dy. \Rightarrow dy \approx f(x) - f(a)$$

On the other hand:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\Downarrow$$

$$\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$$

Because $dx = \Delta x$

$$\Rightarrow \boxed{dy \approx \Delta y}$$

Here, Δy = difference in y.

when the difference in x occurs.

e.g. $\Delta x = x_2 - x_1 \Rightarrow \Delta y = f(x_2) - f(x_1)$

If the function value $f(x_2)$ and $f(x_1)$ are difficult to be calculated, then Δy is difficult to be calculated, in this case,

$dy \approx \Delta y$. i.e.

We use dy to approx. Δy when Δy is hard to be computed.

Ex2: The radius of a sphere was measured and found to be 21 cm with a possible error in the measurement of at most 0.05cm. What is the maximum error in using this value of the

radius to compute the volume of the sphere?

Solution: $V = \frac{4}{3}\pi r^3$, $r = 21\,cm$

but with $\Delta r = 0.05\,cm$.

ΔV according to Δr is:

$$\Delta V \approx dV = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)dr$$

$$= \frac{4\pi}{3} \cdot 3r^2 \cdot dr.$$

$$= 4\pi r^2 \boxed{dr}$$

$$= 4\pi r^2 \boxed{\Delta r}$$

$$= 4\pi r^2 \cdot 0.05 = 4\pi (21)^2 \cdot 0.05$$

$$= 0.2\pi (21)^2 \approx 277\,cm^3$$

Relative error: $\frac{\Delta V}{V} \approx \frac{dV}{V}$

dimensionless

$= \frac{4\pi r^2 \cdot 0.05}{\frac{4}{3}\pi r^3}$ $\Delta r = dr$

$= 3\frac{0.05}{r}$

$= 3\frac{0.05}{2r}$

$= \frac{0.07}{7}$

≈ 0.007

dimensionless

$= 0.7\%$ #