Ch1 Functions & Limits.

§1.1 Functions and their representations.

1: $y = f(x)$ (Algebraic)

① Equation

② x: independent variable. (input value)

Domain of "f": the "home" of x which makes $f(x)$ defined.

③ y: dependent variable: through "f" (output value)

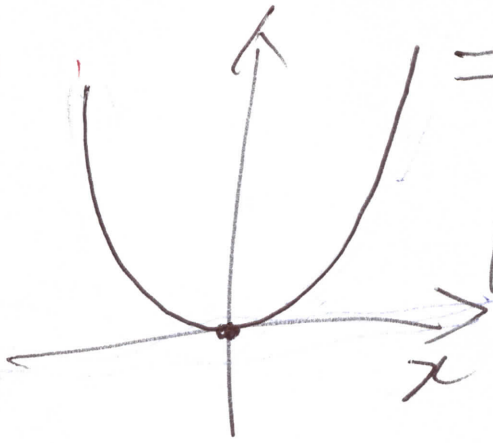
Range of "f": the "home" of "y"

① "f": the relationship between x and y.

"$\in$": in

$x \in \mathbb{R}$

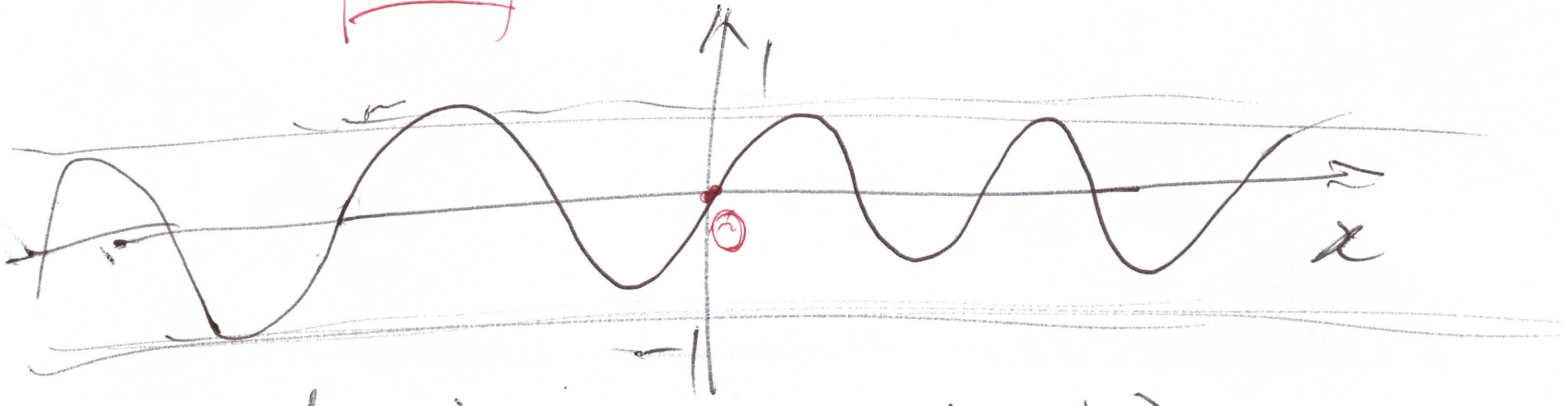
e.g. $y = x^2$



$\Rightarrow$ Domain: $x \in (-\infty, \infty)$

Range: $y \in [0, \infty)$

$y = \sin x$



domain: $x \in (-\infty, \infty)$

Range: $y \in [-1, 1]$.

⑤ Graph: (Geometric)

2: Polynomial of degree "n"

$$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x^1 + a_0$$

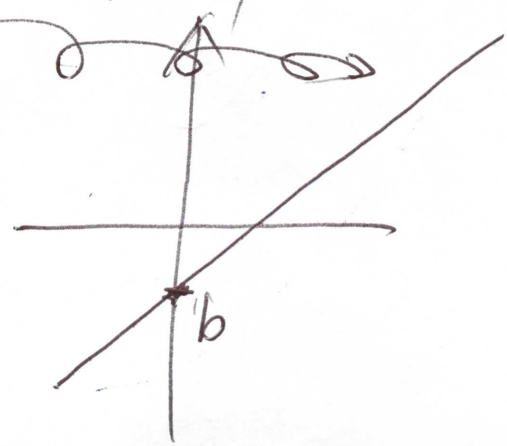
a_i $(i = 0, 1, \cdots, n)$
coefficients

① Linear function (~~pol~~

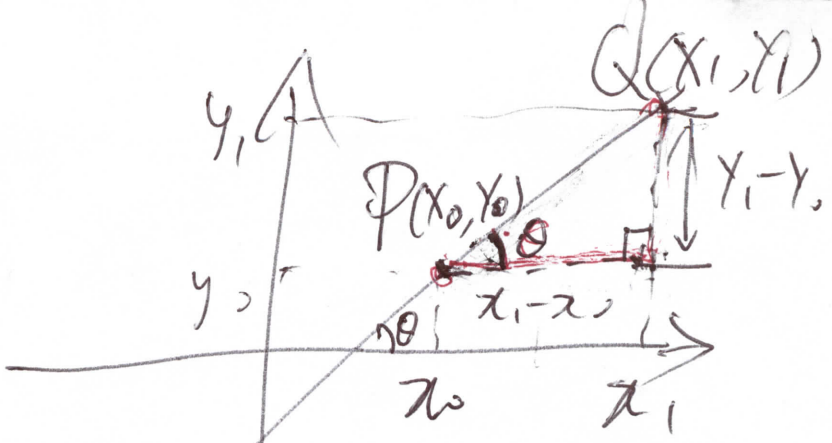
(~~the~~ first degree polynomial)

$$y = ax + b$$

a: slope
b: y intercept



Slope:



$$\text{slope} = \frac{y_1 - y_0}{x_1 - x_0} = \tan\theta$$

Ex1: Find the domain of each function:

(a) $y = f(x) = \sqrt{x+2}$.

$x + 2 \geq 0 \Rightarrow x \geq -2$.

$\Rightarrow$ domain: $[-2, \infty)$

Range: $y \in [0, \infty) \Leftrightarrow y \geq 0$.

(b) $g(x) = \dfrac{1}{x^2 - x}$.

(Rational function) $= \dfrac{P(x)}{Q(x)}$

$P(x)$ and $Q(x)$ are polynomial.

$\therefore x^2 - x \neq 0$.

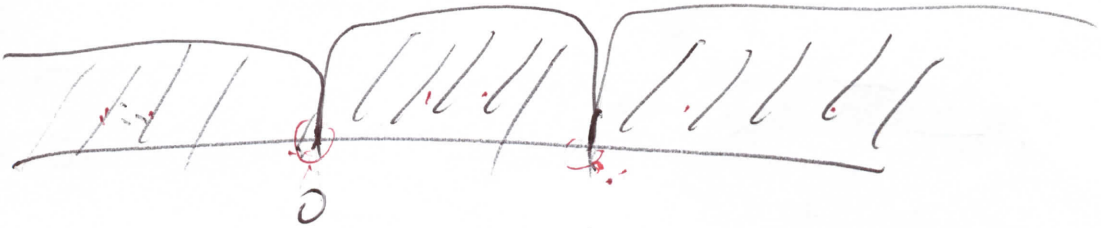
$\Downarrow$

$x(x-1) \neq 0 \Rightarrow$ $x \neq 0$ and $x - 1 \neq 0$

$\Downarrow$

$x \neq 0$ and $x \neq 1$

Domain:



$(-\infty, 0) \cup (0, 1) \cup (1, \infty)$

polynomial, rational functions:

their range are always

$$(-\infty, \infty).$$

the domain of polynomial is also: $(-\infty, \infty)$.

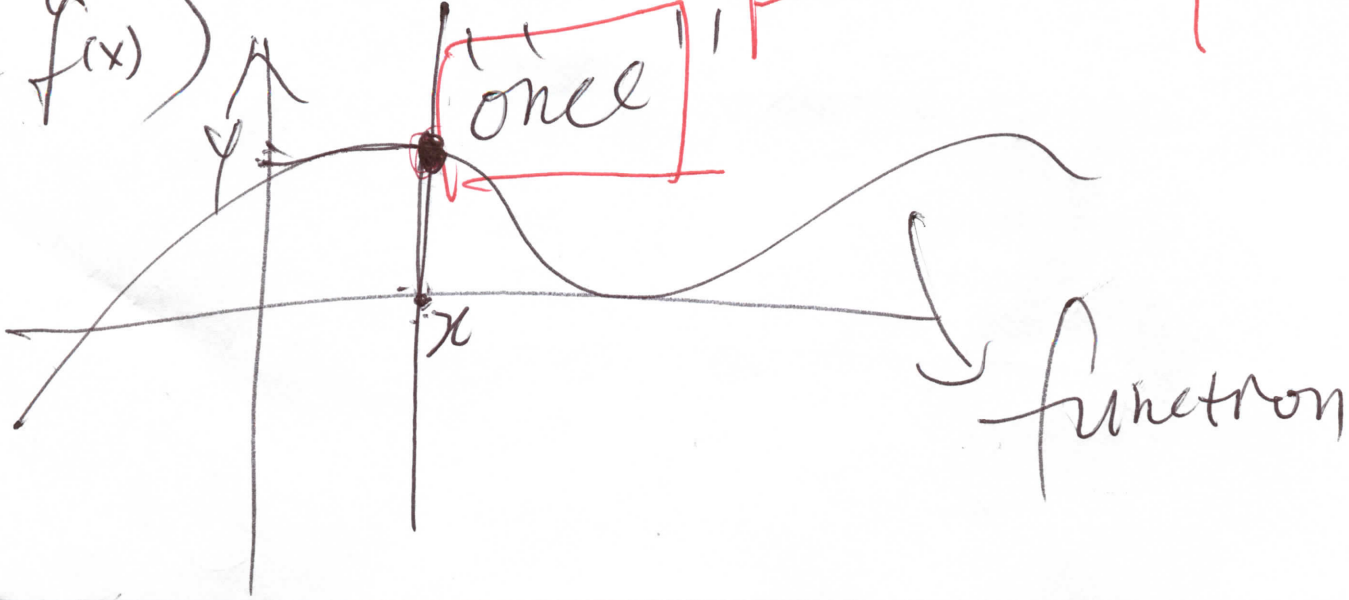
"Set" format "$\in$" ":" "in"

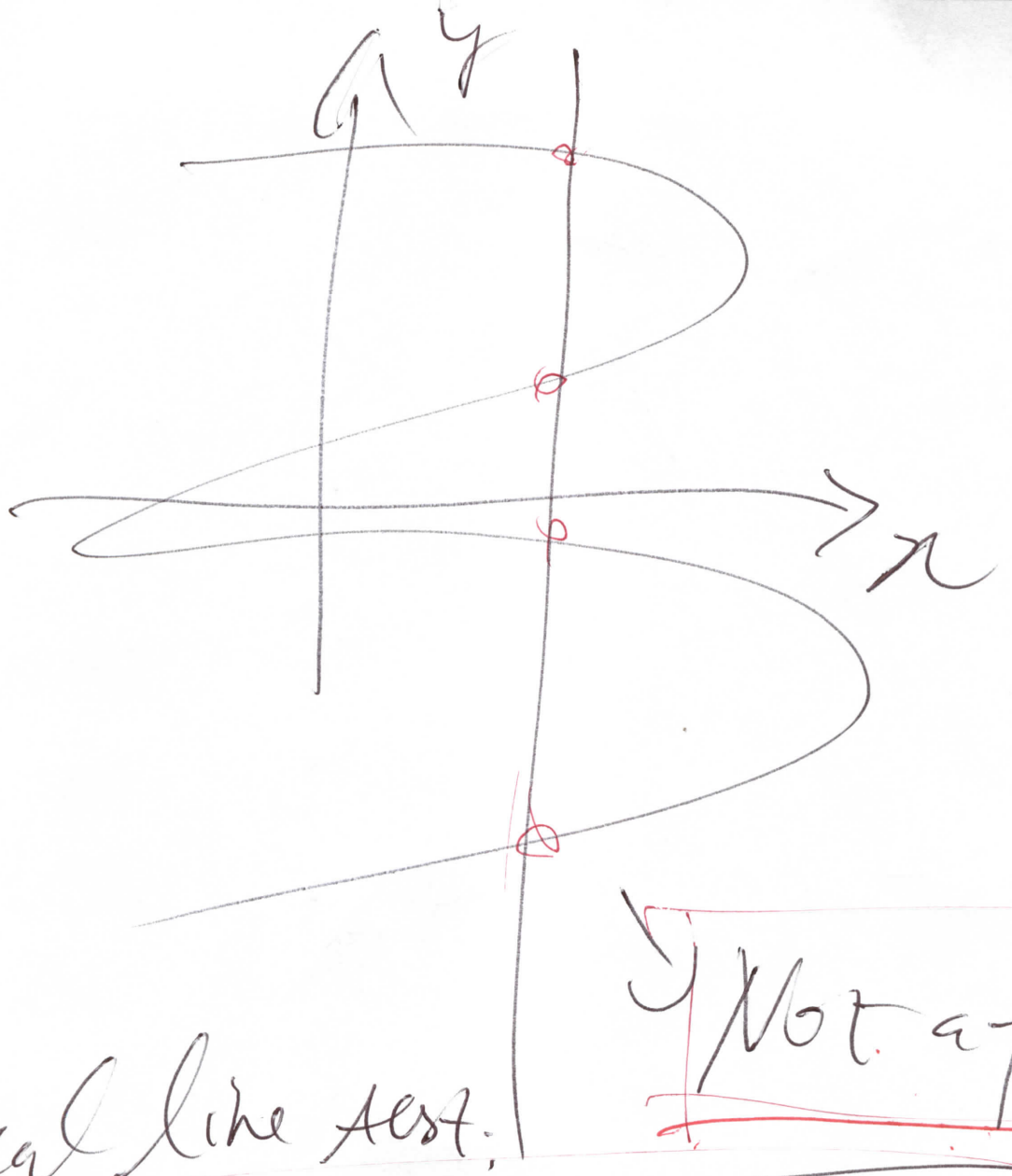
Domain $= \{x \mid x \in R, x \neq 0, x \neq 1\}$

where, $R = (-\infty, \infty)$

3: The vertical line test

→ check if the given equation is a function ($y = f(x)$)





Not a function

Vertical line test:

One-to-one function

The only type of function we use in the "calculus".

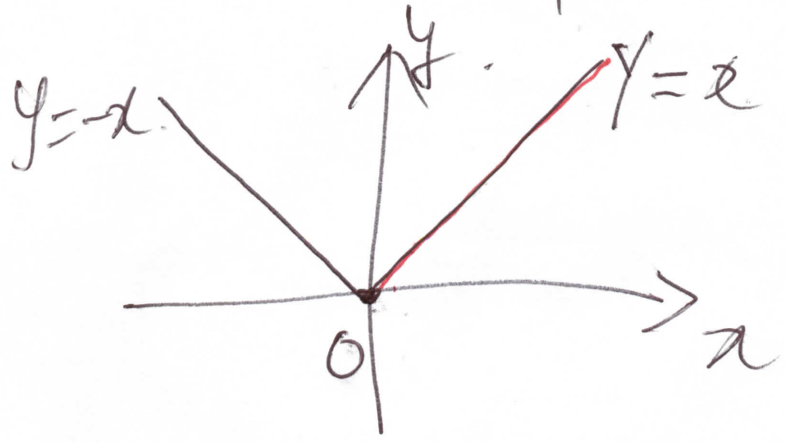
Ex 2: The absolute value function

$y = |x|$ ..., sketch its graph and its domain & Range:

Solution:

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

A piecewise function.



Domain: $(-\infty, \infty)$

Range: $[0, \infty)$.

4: Symmetry of function:

① : Even function:

$f(-x) = f(x)$

eg: $y = x^2$ is even : $(-x)^2 = x^2$



: The graph is symmetric about y-axis.

② Odd function:

$f(-x) = -f(x)$

eg: $y = x^3$ is odd : $(-x)^3 = -x^3$



: The graph is symmetric about the origin.

Ch1: Functions and Limits.

§1.1: Functions.

$$\boxed{y = f(x)}$$: Algebraic expression

① x: independent variable

② y: dependent variable

③ f: relationship of x and y.

④ Domain: x is defined in there.

(5) Range: y-values defined on there.

(6) Graph: it is the geometric interpretation.

§1.2: A catalog of essential functions.

1: Polynomial

① Linear function: (first degree polynomial)

$y = a + bx$: A line.

where, "b" : slope.

"a" : y-intercept.

$-\frac{a}{b}$: x-intercept

by solving $a + bx = 0$ for x.

$$x = -\frac{a}{b}$$

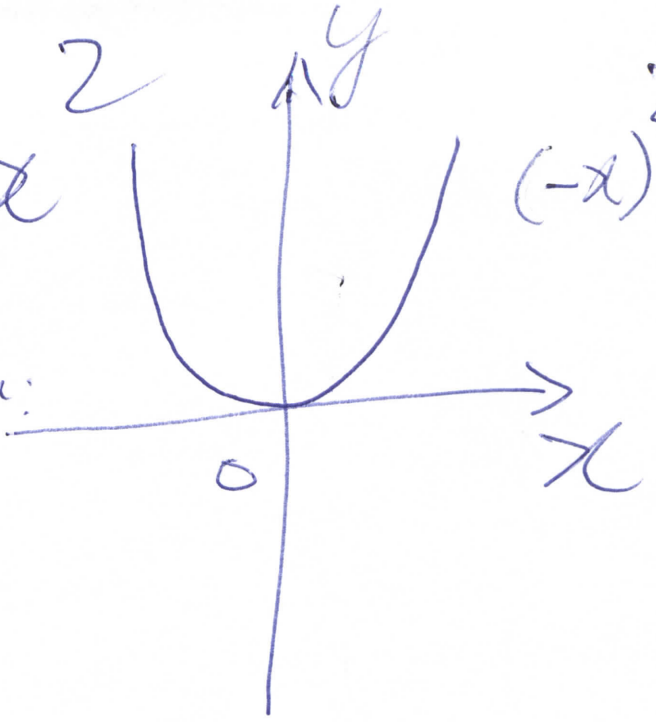
(2) the n-th degree polynomial.

$$y = P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

e.g. $y = x^2$

$(-x)^2 = x^2$

: even function:

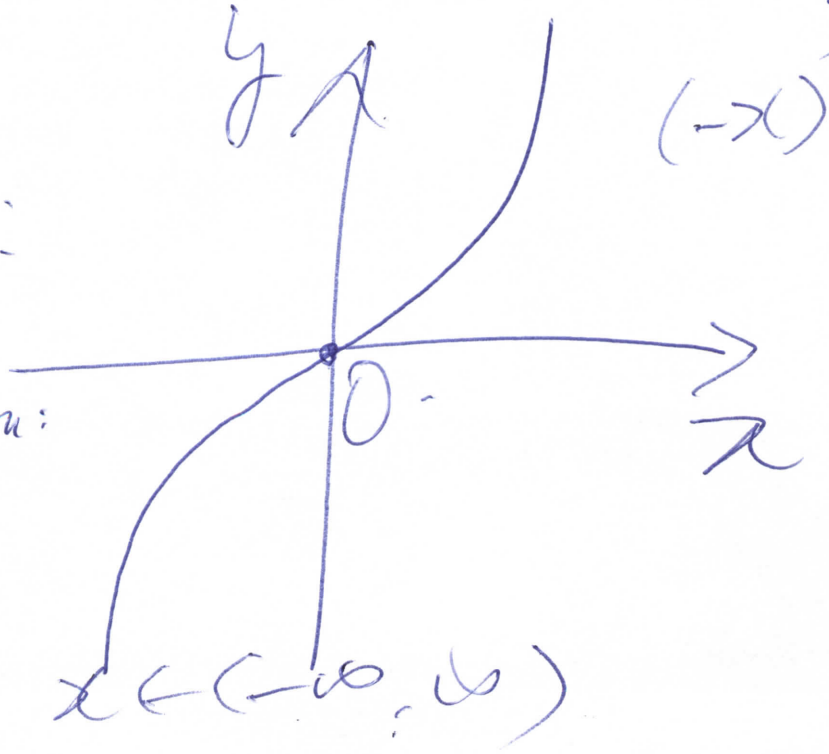


Domain: $x \in (-\infty, \infty)$

Range: $y \in [0, \infty)$

$y = x^3$:

$(-x)^3 = -x^3$



Odd function:

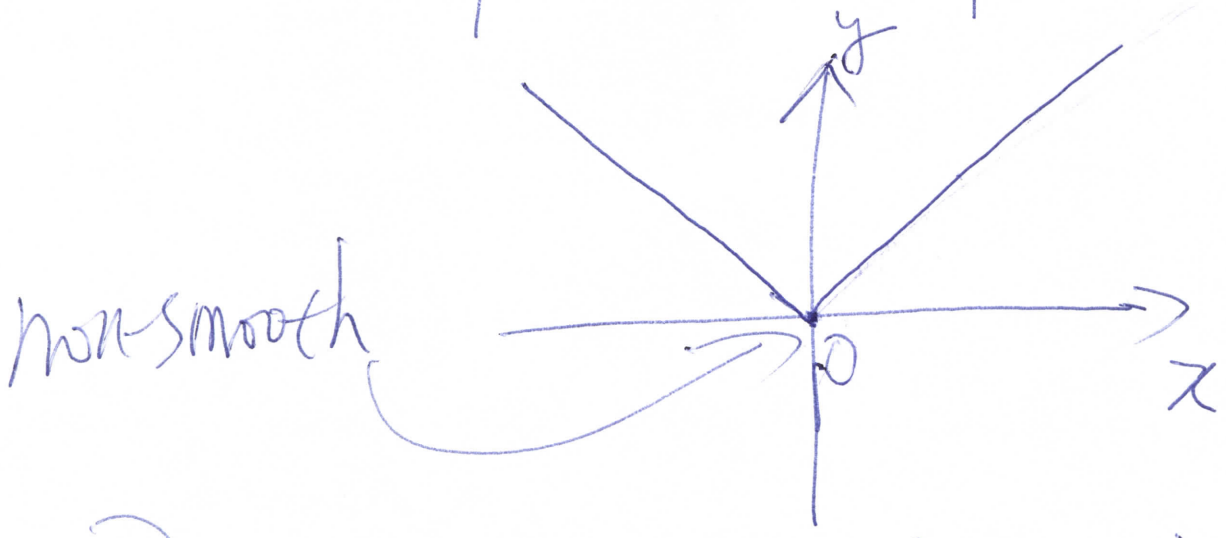
Domain: $x \in (-\infty, \infty)$

Range: $y \in (-\infty, \infty)$.

Eg: $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$

" " "

Piecewise function



∴ Domain : $x \in (-\infty, \infty)$.

Range : $y \in [0, \infty)$.

$y \geq 0$

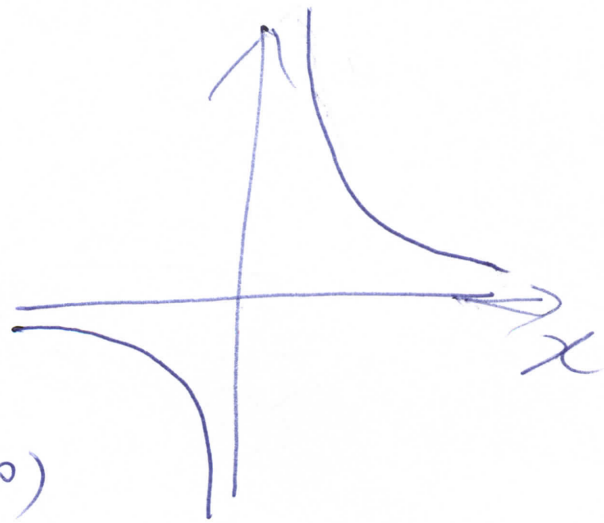
① Even function: $f(x) = f(-x)$.

whose graph is symmetric about the y-axis

② Odd function: $f(-x) = -f(x)$.

whose graph is symmetric about the origin.

③ : $y = \frac{1}{x}$ ⇒



Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

1.2 A Catalog of Essential Functions.

1: Polynomials.

2: Rational Functions.

2.5 : Rational functions.

$$f(x) = \frac{P(x)}{Q(x)}$$

where, P and Q are polynomials.

e.g. $y = \dfrac{x^2 - x + 1}{2x^3 + 4x^2 - 3x + 3}$

Domain: Any value of x in $(-\infty, \infty)$ except those making $Q(x) = 0$.

Range: $(-\infty, \infty)$

5.6: Exponential function

$y = a^x$ $a = "e" = 2.78\ldots$

$\boxed{y = e^x}$ natural exponential

7.4: Logarithmic function

if $a = "e"$

$y = \log_a x = \ln x$

8.5: Trigonometry:

	Domain	Range
$\sin x$	$(-\infty, \infty)$	$[-1, 1]$
$\cos x$	$(-\infty, \infty)$	$[-1, 1]$
$\tan x$	$(-\infty, \infty)$ except $x = \frac{\pi}{2}k$, k is any integer	$(-\infty, \infty)$
$\cot x$	$(-\infty, \infty)$ except $x = k\pi$, k is any integer	$(-\infty, \infty)$
$\sec x$		
$\csc x$		

identities of Trig.

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

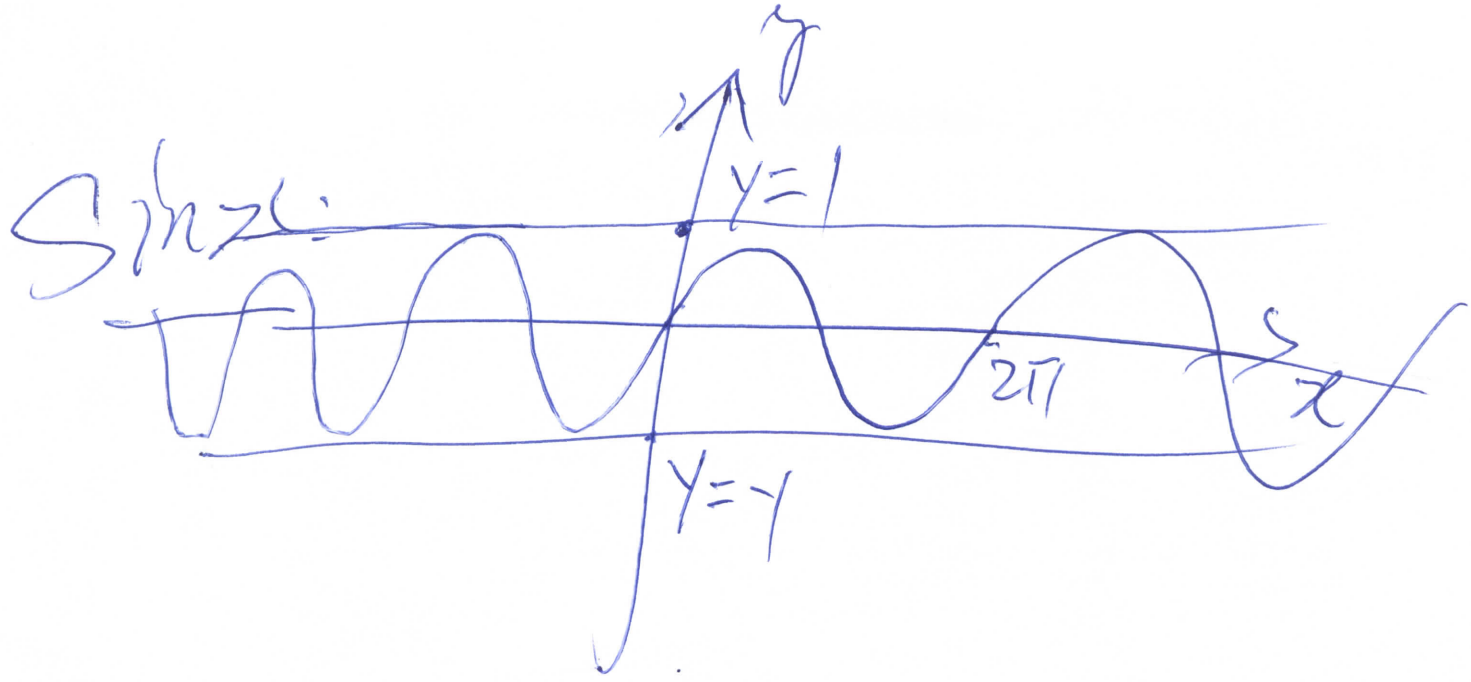
$$\sec x = \frac{1}{\cos x}, \quad \csc = \frac{1}{\sin x} \quad \tan x = \frac{1}{\cot x}.$$

$$\sin^2 x + \cos^2 x = 1 \leftarrow \text{Pythagorean identities}$$

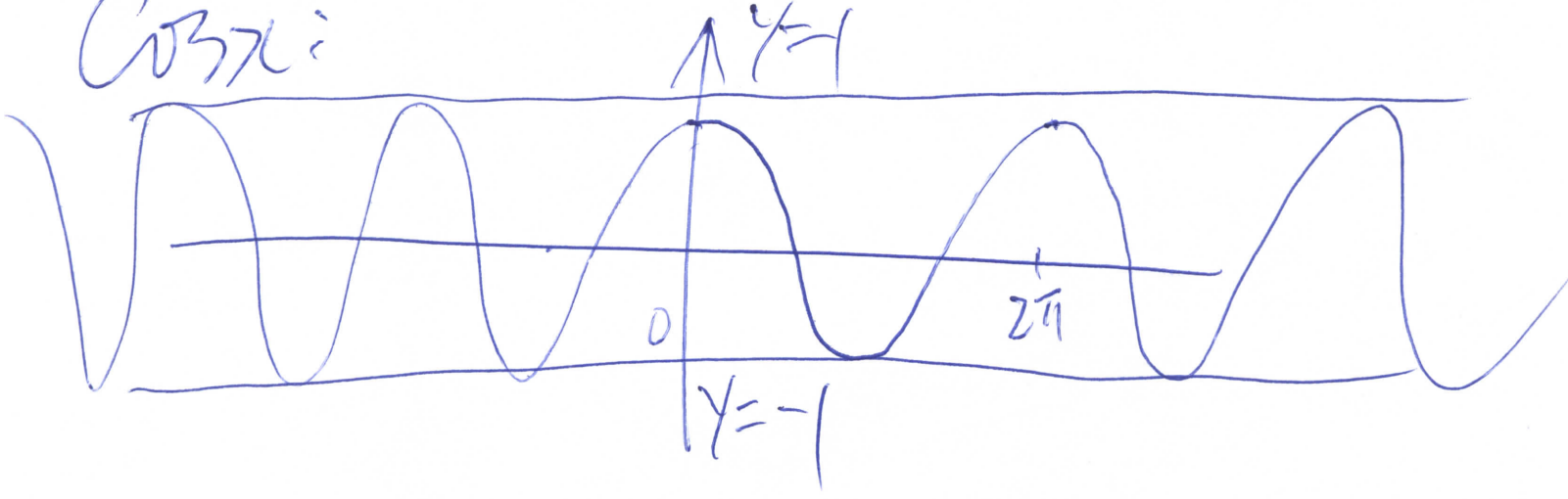
$$\tan^2 x + 1 = \sec^2 x$$

$$\cot^2 x + 1 = \csc^2 x.$$

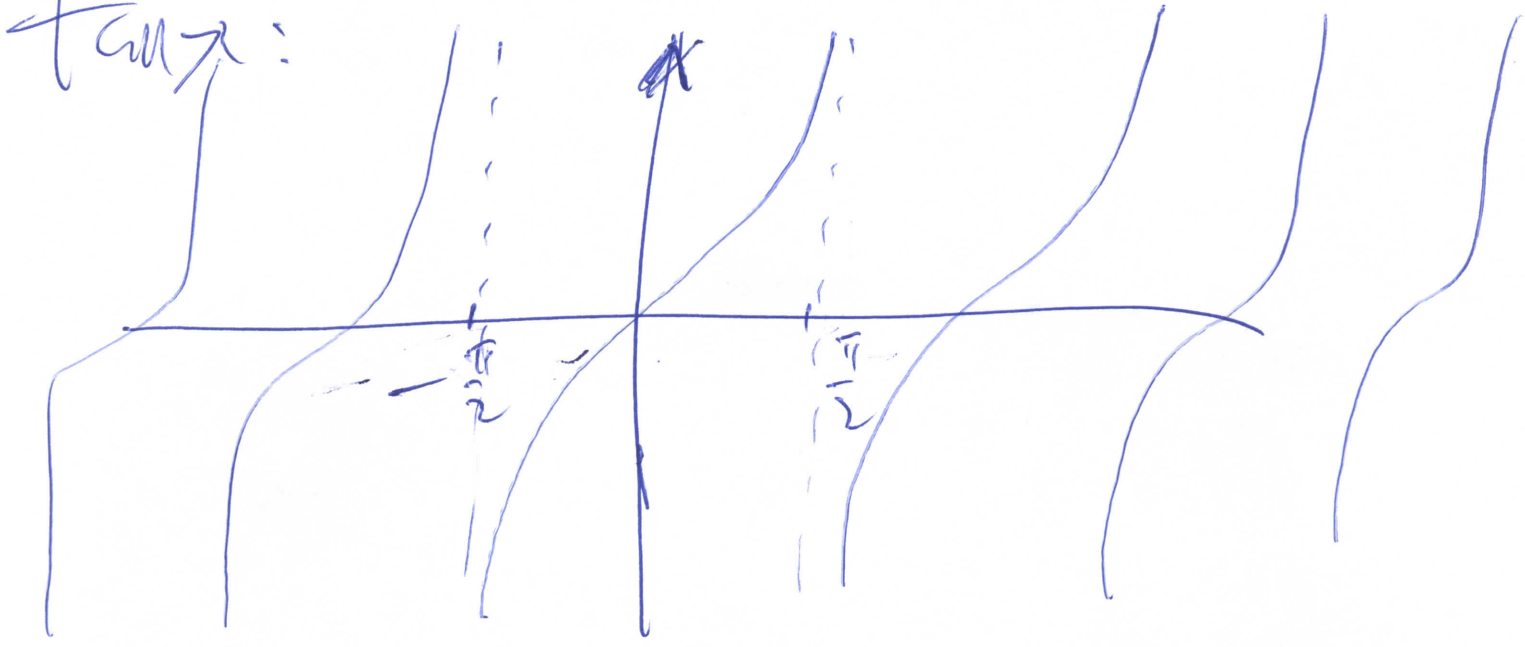
Sinx:



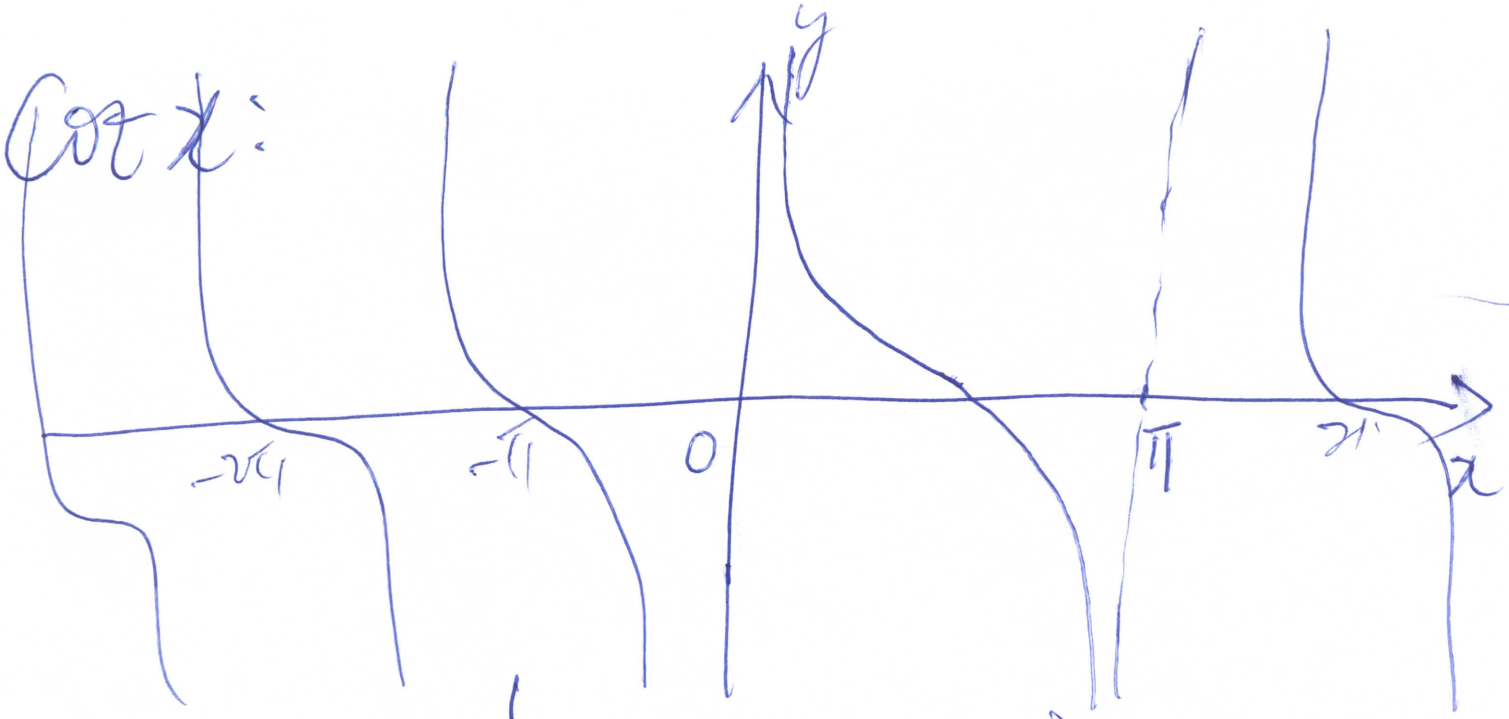
Cosx:



tanx:



Cot x:



$$\sec x = \frac{1}{\cos x},$$

$$\csc x = \frac{1}{\sin x},$$

$$\tan x = \frac{\sin x}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x.$$

$$1 + \cot^2 x = \csc^2 x.$$

1.6: Composite function:

$$f \circ g(x) = f(g(x))$$

$$f \circ g \circ h(x) = f(g(h(x)))$$

e.g. $\sqrt{2-x} = \sqrt{g(x)}$, where

Two-layer.

$g(x) = 2 - x$

and $f(u) = \sqrt{u}$.

Let $u = g(x) = 2 - x$

$\Rightarrow \sqrt{2-x}$

$\sqrt{2-\sqrt{x}}$: $f(x) = \sqrt{g(x)}$

$2 - h(x) = g(x) = \sqrt{h(x)}$

$h(x) = \sqrt{x}$

$\Rightarrow \sqrt{2-\sqrt{x}}$

Ex1: Given $f(x)=\sqrt{x}$,

$g(x)=\sqrt{2-x}$ ~~$\neq f(x)$~~

find $g\circ f$ and its domain.

Solution: $g\circ f(x)=g(f(x))$

$=\sqrt{2-f(x)}$

$=\sqrt{2-\sqrt{x}}$.

$f\circ g(x)=f(g(x))$

$=\sqrt{g(x)}$

$=\sqrt{\sqrt{2-x}}=\sqrt{(2-x)^{\frac{1}{2}}}$

$=\left((2-x)^{\frac{1}{2}}\right)^{\frac{1}{2}}=(2-x)^{\frac{1}{4}}$

$\neq\sqrt{2-x}$.

Find the domain of $\sqrt{2-\sqrt{x}}$:

$\sqrt{x} \Rightarrow x \geq 0 \Rightarrow x \in [0, \infty)$

$\sqrt{2-\sqrt{x}} \Rightarrow 2-\sqrt{x} \geq 0 \Rightarrow (\sqrt{x})^2 \leq (2)^2$

$\Rightarrow x \leq 4 \Rightarrow x \in (-\infty, 4]$

$\Rightarrow 0 \leq x \leq 4$

$\Rightarrow \boxed{x \in [0, 4]}$

$f \circ g(x)$

Find the domain of $\sqrt{\sqrt{2-x}}$.

Let $2-x \geq 0 \Rightarrow \boxed{x \leq 2}$

$\boxed{(-\infty, 2]}$

to make $\sqrt{2-x}$ is defined.

$\sqrt{\boxed{\sqrt{2-x}}}$ is well defined then.

§1.3 The limit of a function

Ex 1: Investigate $f(x) = x^2 - x + 2$ for values of x near "2".

$1 = 1 + 2 = 2$

"near" "2" $\Leftrightarrow x \neq 2$

Solutions:

x	$f(x)$	x	$f(x)$
1.	2.	3.	8.
1.5	2.75	2.5	5.25
1.8	3.44	2.2	4.04
1.9	3.71	2.1	4.31
1.95	3.8525	2.05	4.1525
1.99	3.985025	2.01	4.0301
1.999	3.997001	2.005	4.020
		2.001	4.002001
2.	4.	2.	4.

Left side of "2"

right side of "2"

$f(x) = x^2 - x + 2 \longrightarrow 4$ as $x \to 2$ from both sides

1: Definition of limit:

$$\lim_{x \to a} f(x) = L$$

finite real number.

Read as: The limit of $f(x)$, as x approaches "a" "from both sides" equals "L".

where, L is a finite number.

then, we call $f(x)$ exists the limit that is "L" as $x \to a$.

Remark: If $\lim_{x \to a} f(x) = \pm\infty$

$\Rightarrow$ $f(x)$ does NOT exist the limit

Remark: ① If we can make the values of $f(x)$ arbitrarily close to "L" (as close as we like) by taking x to be sufficiently close to "a" on either side of "a", but NOT equal to "a"

② "$x \neq a$" $\Rightarrow$ we never consider $x = a$ in finding the limit of $f(x)$ as $x \to a$.

$\Leftrightarrow$ x is near "a"

~~10~~

Step 1: $\left(\frac{1}{2}\right)^1$

~~Step~~ 2: $\frac{1}{4} = \left(\frac{1}{2}\right)^2$

⋮

Step n: $\left(\frac{1}{2}\right)^n \longrightarrow 0$ as $n \to \infty$.

lim, 7.

$n \to \infty$.

$\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

Limit: A motion principle

☆ moving process.

Step 1: "$\frac{1}{2}$" is remained

Step 2: $\left(\frac{1}{2}\right)^2$ is remained

Step 3: $\left(\frac{1}{2}\right)^3$ is remained

$\vdots$

Step n: $\left(\frac{1}{2}\right)^n$ is remained

$n \to \infty$: $\frac{1^n}{2^n} = \frac{1}{2^n} = \frac{1}{\infty} = 0$

l : The remaining dist.

$$\frac{1}{2}\text{"}l\text{"} = \frac{l}{2}$$

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} l = 1$$

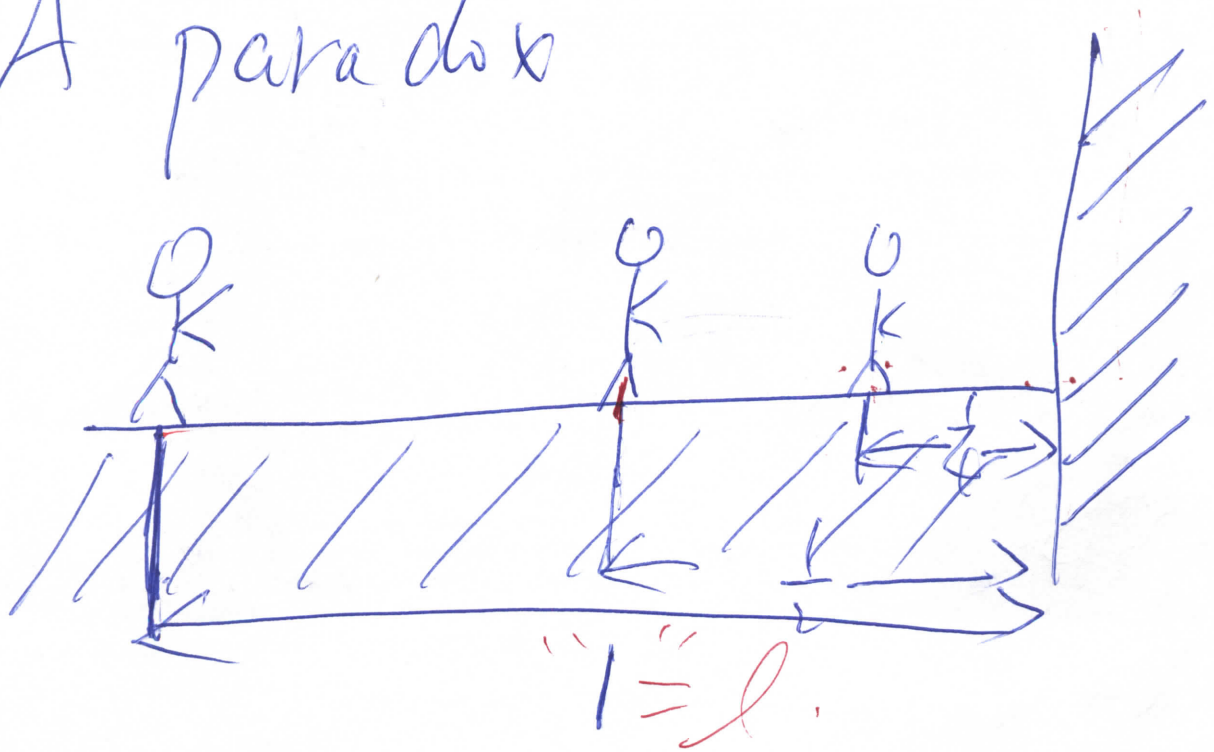
The remaining dist. $= \left(\frac{1}{2}\right)^n$ n : # of step

$$\left(\frac{1}{2}\right)^n \to 0 \quad \text{as } n \to \infty$$

$$= \frac{1^n}{2^n} = \frac{1}{2^n} = \frac{1}{\infty} = 0$$

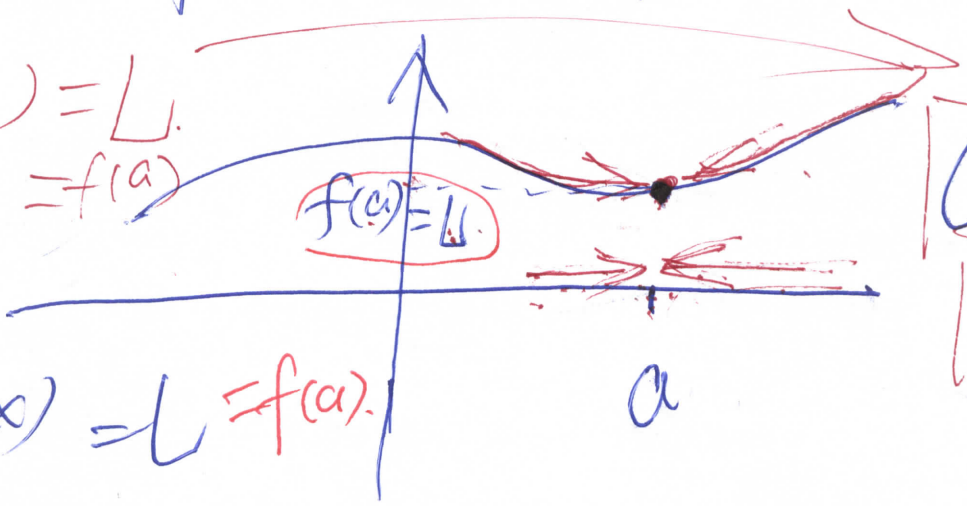
③ In fact. $f(x)$ needs not. even be defined when $x = a$ or: $f(a)$ may not exist. The only thing that matters is how $f(x)$ is defined near when x "a".

"A paradox"



④ Let $f(a)$ is defined and $f(a)=L$

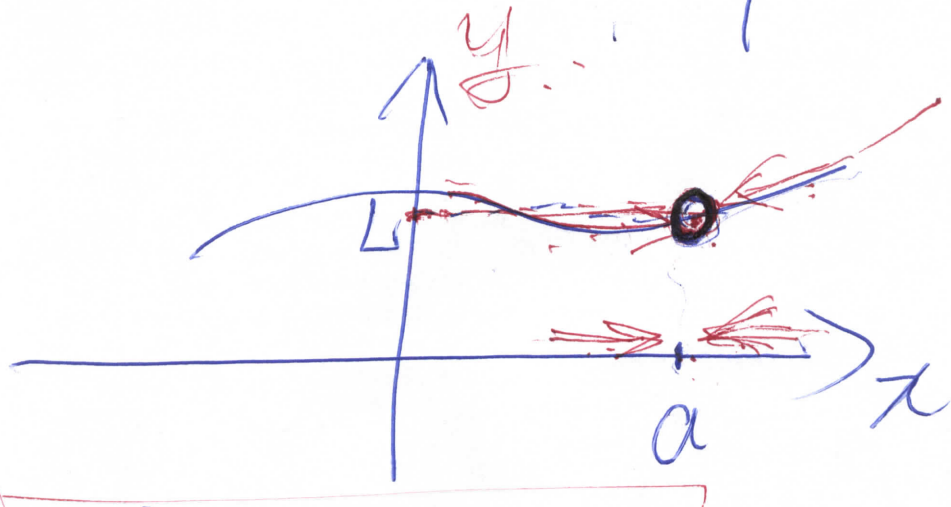
$\lim_{x\to a} f(x) = L = f(a)$



Continuous

§1.5

⑤ Let $f(a)$ is undefined.

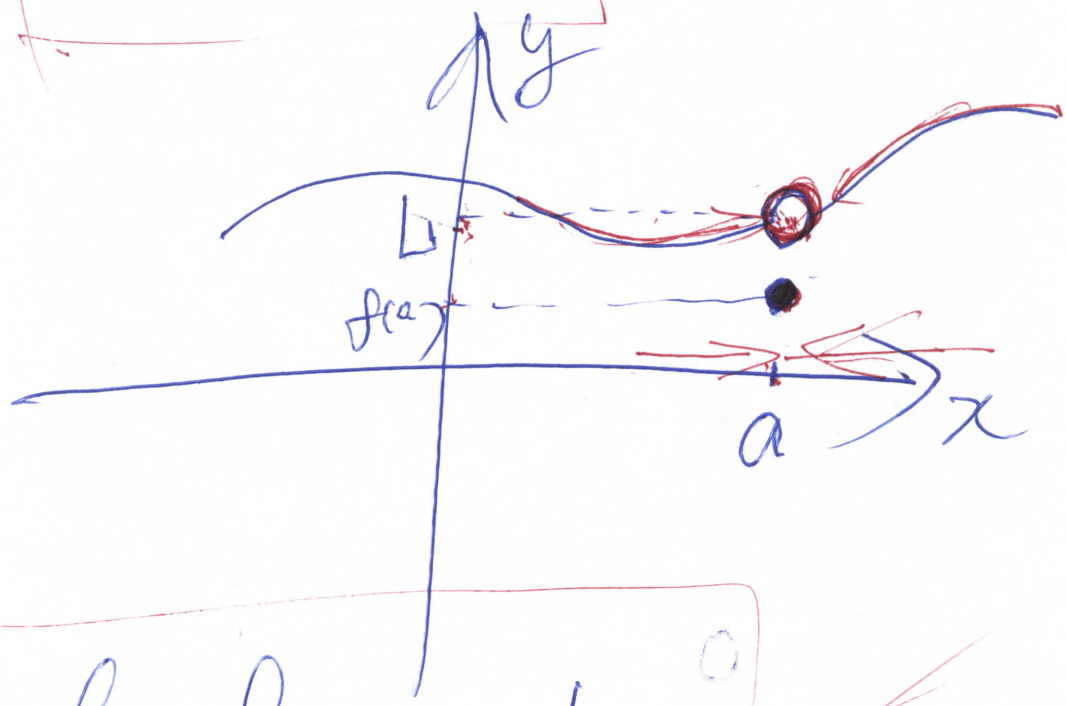


$$\lim_{x\to a} f(x) = L$$

§1.5: Discontinuous Case I.

(6) Let $f(a)$ is defined but

$$f(a) \neq L.$$



$$\lim_{x \to a} f(x) = L.$$

Discontinuous case 2.

2: One-Sided Limit

① $\lim_{x \to a^-} f(x) = L$: ~~the left~~

$x \to a^-$ → x approaches a from the left

the left-hand limit of $f(x)$ $(x < a)$

② $\lim_{x \to a^+} f(x) = L$:

$x \to a^+$ → x approaches a from the right

the right-hand limit of $f(x)$.

$(x > a)$

3: Theorem of Limit

$\lim_{x \to a} f(x) = L$ exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

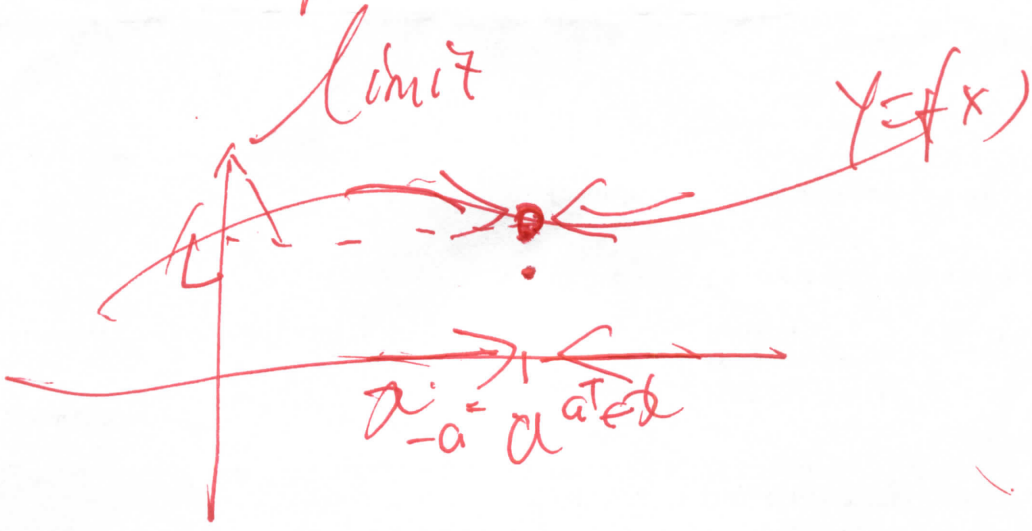
Theorem of Limit

$\lim_{x \to a} f(x) = L$ if and only if

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$

Left-hand limit

Right-hand limit



If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

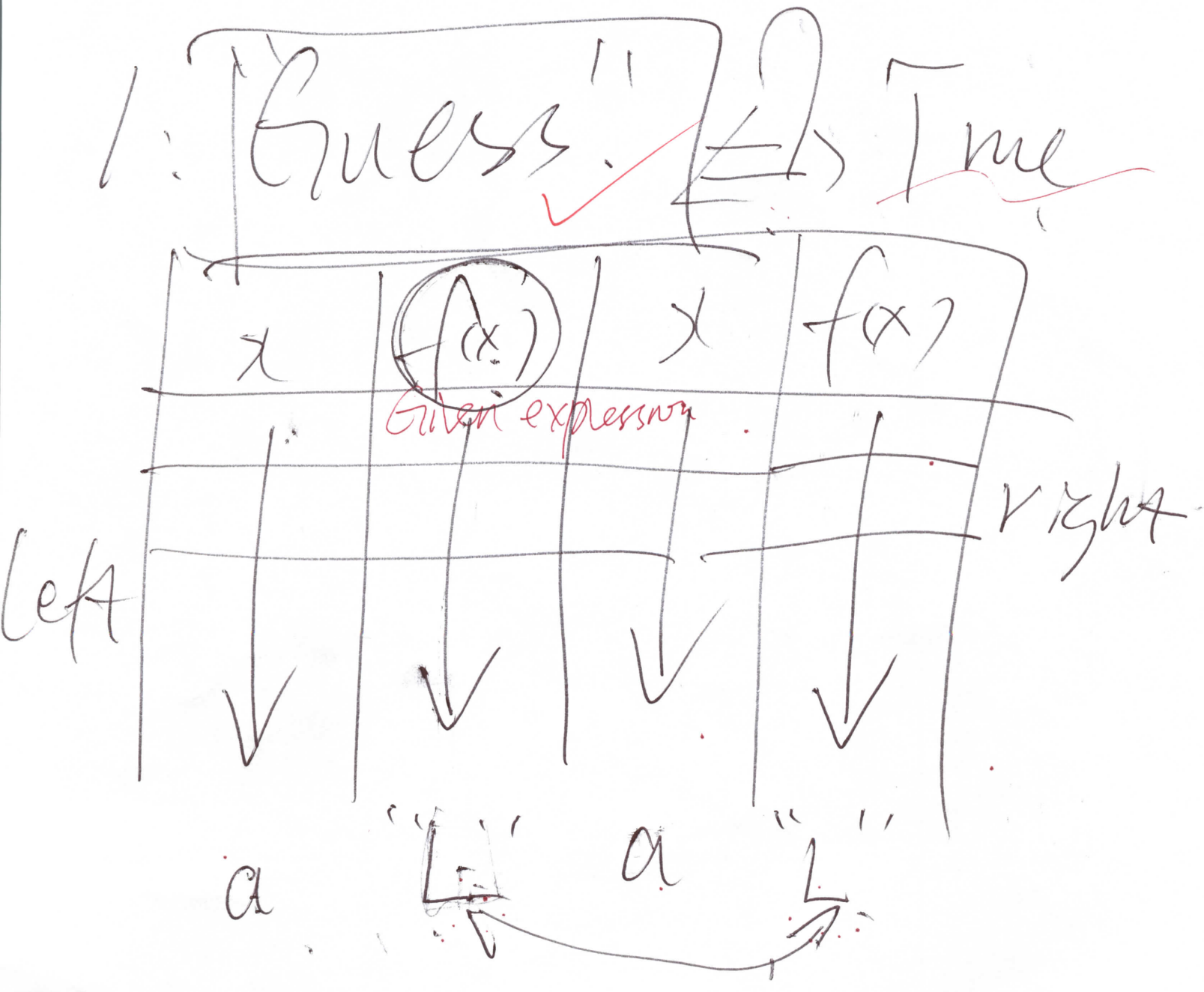
then $f(x)$ does not exist limit.

If $\lim_{x \to a} f(x)$ does not exist.

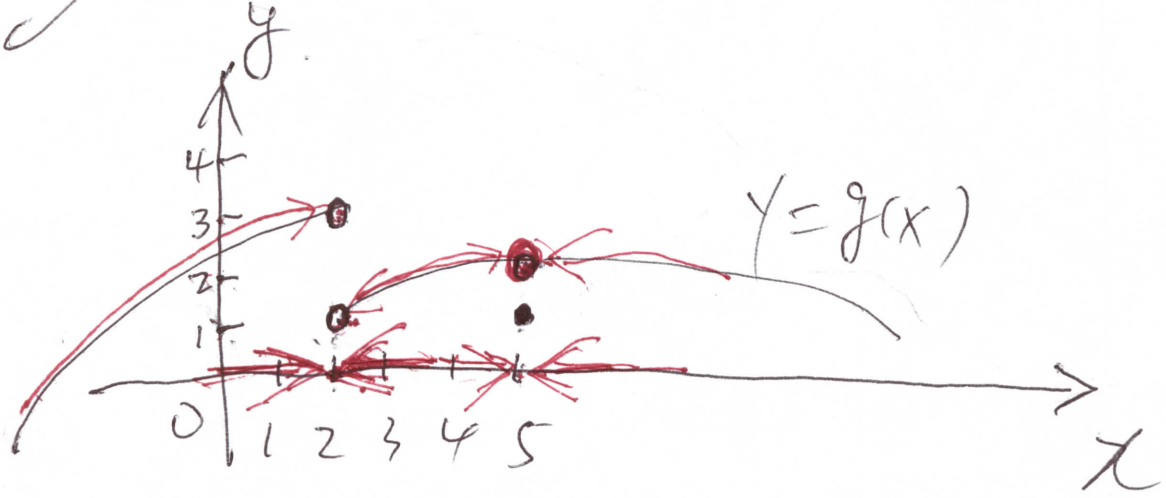
$$\lim_{x \to a} f(x) = L$$

How to evaluate the limit

1: "Guess" $\Leftrightarrow$ Time



Ex: Function $g(x)$ is shown in the figure. Use it to state the values of the following limits:



(a) $\lim_{x \to 2^-} g(x) = 3$

(b) $\lim_{x \to 2^+} g(x) = 1$

(c): $\Rightarrow \lim_{x \to 2} g(x) =$ does not exist.

(d) $\lim_{x \to 5^-} g(x) = 2$

(e) $\lim_{x \to 5^+} g(x) = 2$

$\lim_{x \to 2^-} g(x) = 3$

$\lim_{x \to 2^+} g(x) = 1$

$\Big\}$ $\lim_{x \to 2} g(x)$ does not exist

$\lim_{x \to 5^-} g(x) = 2$

$\lim_{x \to 5^+} g(x) = 2$

$\Big\}$ $\therefore \lim_{x \to 5} g(x) = 2$

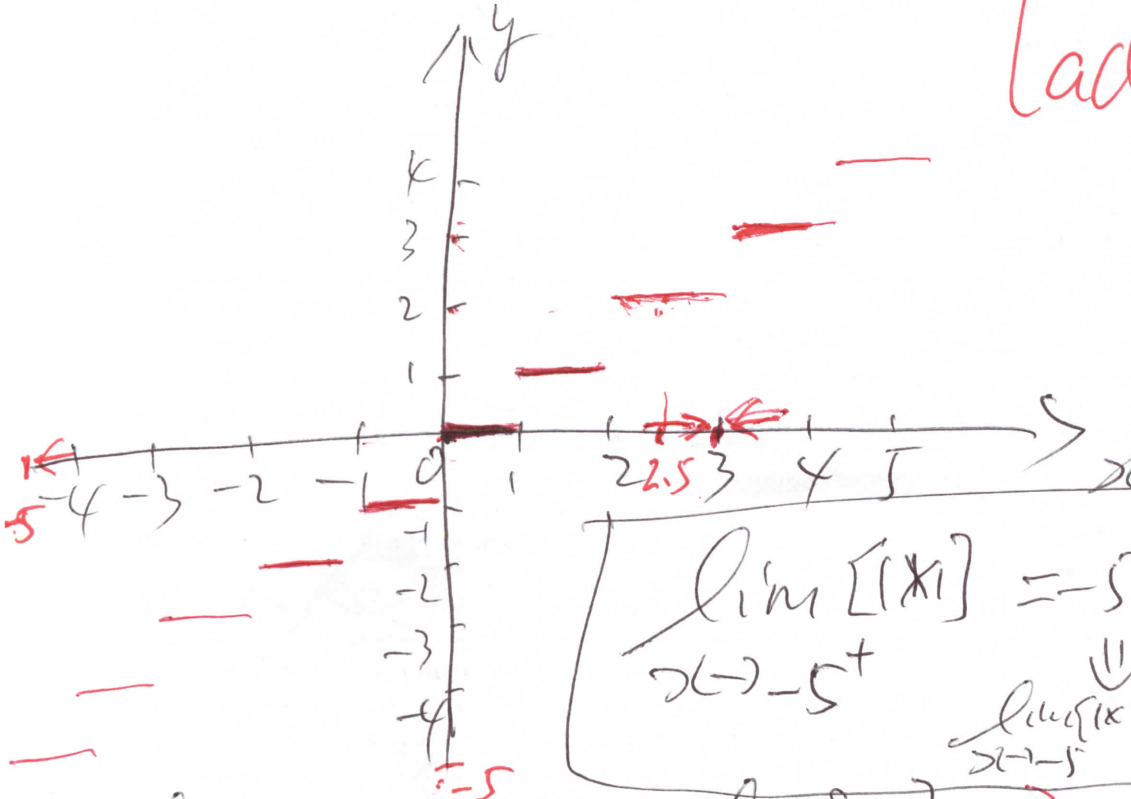
Ex4: The **greatest integer function**

$y = [\![x]\!]$: the greatest integer that is less than or equal x.

e.g.: $[\![5.5]\!] = 5$; $[\![6.9]\!] = 6$

$[\![5.1]\!] = 5$; $[\![-2.2]\!] = -3$

$[\![-10.8]\!] = -11$.



ladder func.

$\lim_{x \to -5^+} [\![x]\!] = -5$, but $\lim_{x \to -5} [\![x]\!] = -6$

$\Downarrow$

$\lim_{x \to -5} [\![x]\!]$ D.N.E.

$\lim_{x \to 3^-} [\![x]\!] = 2$, $\lim_{x \to 3^+} [\![x]\!] = 3$ $\Rightarrow$ $\lim_{x \to 3} [\![x]\!]$ does not exist.

$y = [|x|] \leq x$. and it is an integer.

$[|-2.2|] = -3$.

0.05

$0 \leq x \leq 1 \Rightarrow [|x|] = 0$

$1 \leq x \leq 2 \Rightarrow [|x|] = 1$

(1.3)

$2 \leq x \leq 3 \Rightarrow [|x|] = 2$

$3 \leq x \leq 4 \Rightarrow [|x|] = 3$.

$-1 \leq x \leq 0 \Rightarrow [|x|] = -1$

-0.5

$-2 \leq x \leq -1 \Rightarrow [|x|] = -2$.

⋮ ⋮

$$\lim_{x \to 3^-} [|x|] = 2.$$

$$\lim_{x \to -5^+} [|x|] = -5$$

$-5 \le x \le -4. \Rightarrow [|x|] = -5$

$$\lim_{x \to k} [|x|] \text{ does not exist.}$$

k is an integer

§1.4: Calculating Limits.

1: Limit Laws.

① Sum law:

$$\lim_{x \to a}(f+g) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

② Difference law:

$$\lim_{x \to a}(f-g) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

③ Product law:

$$\lim_{x \to a}(fg) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

④ Quotient law:

$$\lim_{x \to a} \frac{f}{g} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

where $\lim_{x \to a} g(x) \neq 0$

⑤ Constant multiple law:

$$\lim_{x \to a} [C f(x)] = C \lim_{x \to a} f(x)$$

C → constant

⑥ Power law:

$$\lim_{x \to a} [f(x)]^n = \left(\lim_{x \to a} f(x)\right)^n$$

where n is integer.

⑦ Root law:

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

$\sqrt[n]{f(x)} \downarrow (f(x))^{\frac{1}{n}}$.

where $n > 0$ is an integer.

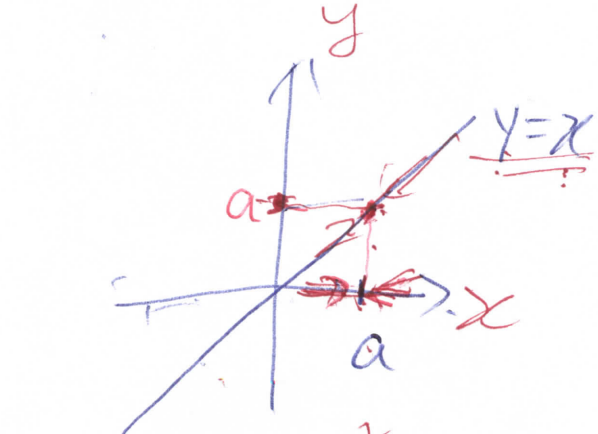
if n is even, then $\lim_{x\to a} f(x) \geq 0$.

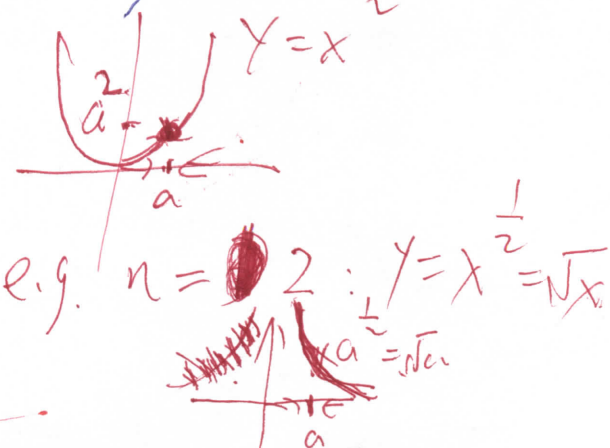
⑧ $\lim_{x\to a} C = C$. (C: constant)

⑨ $\lim_{x\to a} x = a$

⑩ $\lim_{x\to a} x^n = a^n$

⑪ $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$





Ex1: $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$ rational function.

$$= \frac{\lim_{x\to -2}(x^3+2x^2-1)}{\lim_{x\to -2}(5-3x)}$$

$$= \frac{\lim_{x\to -2}x^3 + \lim_{x\to -2}2x^2 - \lim_{x\to -2}1}{\lim_{x\to -2}5 - \lim_{x\to -2}3x} \quad \text{law (8)}$$

law (10) $\frac{(-2)^3 + 2\lim_{x\to -2}x^2 - 1}{5 - 3\lim_{x\to -2}x}$ law (5)

law (8) $= \frac{(-2)^3+2\cdot(-2)^2-1}{5-3\cdot(-2)}$ law (9).

$$= \frac{-8+8-1}{5+6} = \boxed{-\frac{1}{11}} \quad \#$$

If we calculate the function value at $x = -2$:

$$f(-2) = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

$$\Rightarrow \boxed{\lim_{x \to a} f(x) = f(a)}$$

where, we must make sure $\lim_{x \to a} f(x)$ exists, and $f(x)$ is defined at $x = a$.

2: Direct Substitution Property

$$\lim_{x \to a} f(x) = f(a) \quad \text{(D.S.P.)}$$

where, $f(a)$ is defined

if $f(x)$ is a polynomial or

a rational function in its domain

Ex 2: $\lim_{t \to 0} \frac{(\sqrt{t^2+9} - 3)}{t^2} \cdot \frac{(\sqrt{t^2+9} + 3)}{\sqrt{t^2+9} + 3}$

"Introducing something extra" ☆

$$(a+b)(a-b) = a^2 - b^2$$

$$= \lim_{t\to 0} \frac{(\sqrt{t^2+9})^2 - 3^2}{t^2(\sqrt{t^2+9}+3)}$$ ← the key step

Simplification

$$= \lim_{t\to 0} \frac{t^2+9-9}{t^2(\sqrt{t^2+9}+3)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\lim_{t\to 0}\sqrt{t^2+9}+3} = \frac{1}{\sqrt{\lim_{t\to 0}(t^2+9)}+3}$$

D.S.P

$$= \frac{1}{\sqrt{\left(\lim_{t\to 0} t\right)^2+9}+3}$$

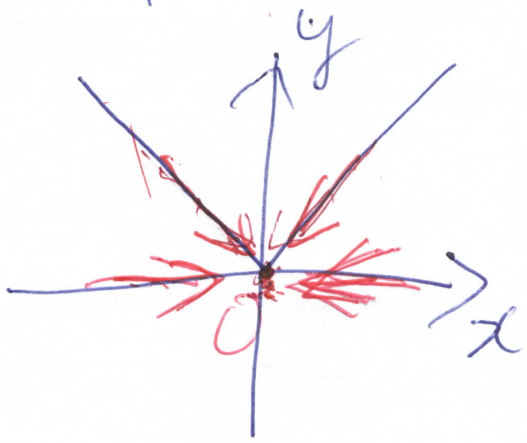
$$= \frac{1}{\sqrt{0+9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$$

#

Ex3: Evaluate $\lim_{x\to 0} |x| = 0$.

Solution:

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



$$\lim_{\substack{x\to 0^- \\ \text{"}x<0\text{"}}} |x| = \lim_{x\to 0^-} (-x) \overset{\text{D.S.P.}}{=} 0 = 0$$

$$\lim_{\substack{x\to 0^+ \\ \text{"}x>0\text{"}}} |x| = \lim_{x\to 0^+} x = 0$$

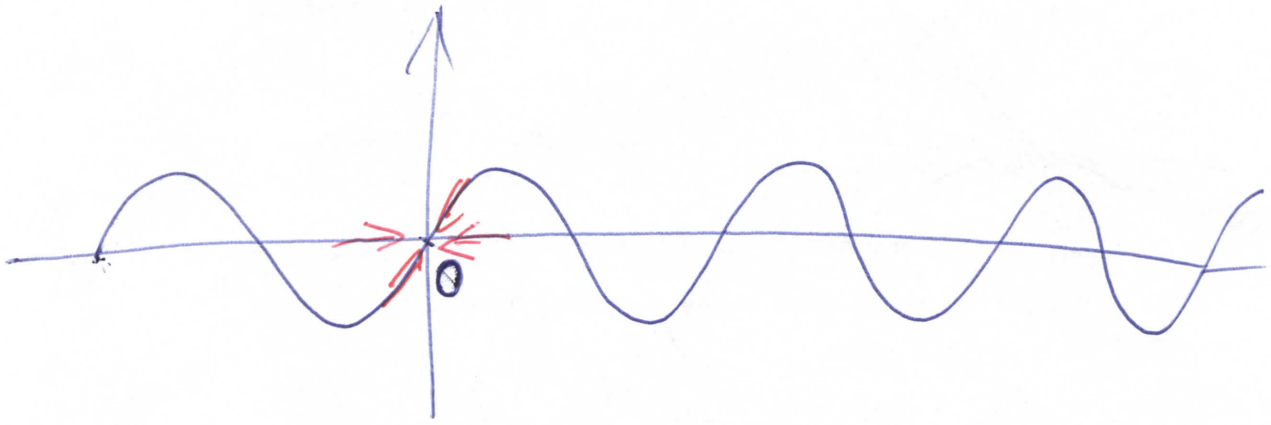
$$\Rightarrow \lim_{x\to 0} |x| = 0 = |0| \quad \#$$

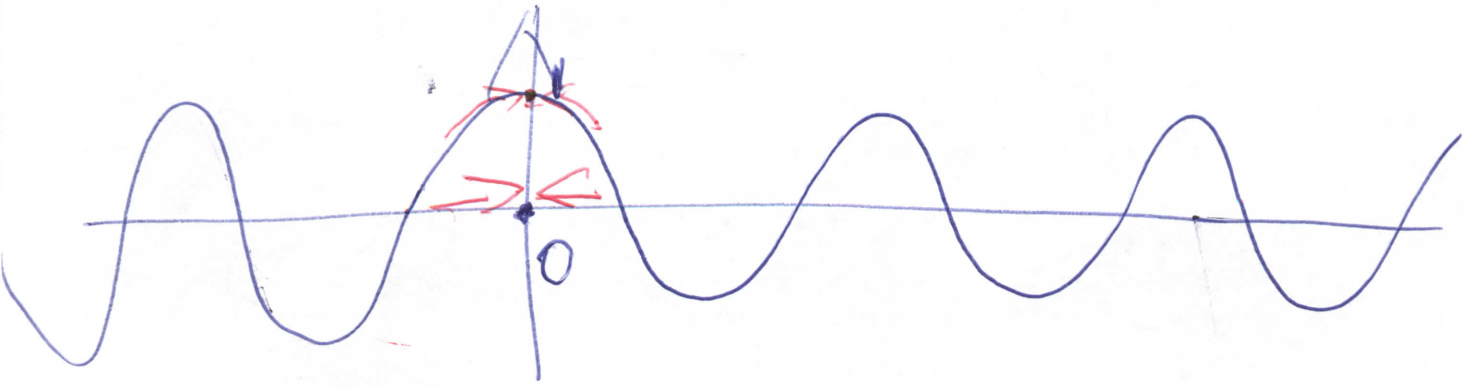
$|x|$ is continuous at "0"

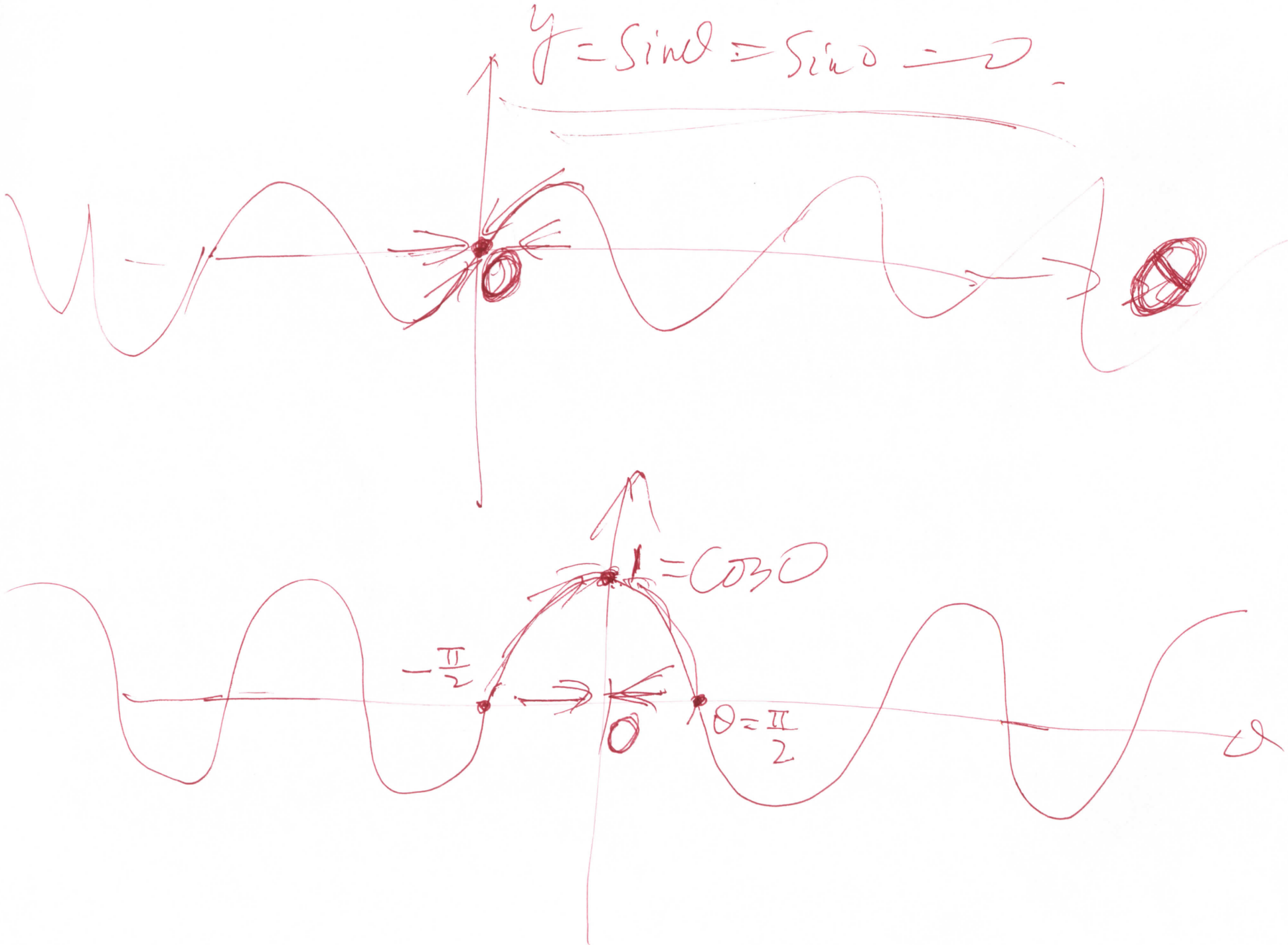
3: $\lim_{\theta \to 0} \sin\theta = 0 = \sin 0$.

"law"

$\lim_{\theta \to 0} \cos\theta = 1 = \cos 0$






y = sinθ = sin0 →
= cos0
$-\frac{\pi}{2}$
0
$\theta = \frac{\pi}{2}$
θ

All limit laws (S.D.S.P) can be applied to one-sided limits as well.

Ex 2: Guess the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$ ☆

Solution: Draw the table

x	$\frac{\sin x}{x}$	x	$\frac{\sin x}{x}$
	.	1	.
-1		0.5	
-0.5			
-0.25		0.25	
-0.1		0.1	
-0.01		0.01	
-0.001		0.001	
-0.0001		0.0001	
-0.00001		0.00001	
↓	↓	↓	↓
0	1	0	1

Ex 4: $\lim_{\theta \to 0} \frac{\cos 3\theta - 1}{\theta}$

Solution 1: $= \frac{\lim_{\theta \to 0} \cos 3\theta - 1}{\lim_{\theta \to 0} \theta}$

$= \frac{1 - 1}{0} = \frac{0}{0}$

"indeterminant form"

Solution 2: $\lim_{\theta \to 0} \frac{(\cos 3\theta - 1)(\cos 3\theta + 1)}{\theta (\cos 3\theta + 1)}$

$= \lim_{\theta \to 0} \frac{\cos^2 3\theta - 1}{\theta(\cos 3\theta + 1)} = \lim_{\theta \to 0} \frac{-\sin^2 3\theta}{\theta(\cos 3\theta + 1)}$

$$= -\lim_{\theta \to 0} \left(\frac{\sin\theta}{\theta} \right) \left(\frac{\sin\theta}{\cos\theta + 1} \right)$$

$$= -\left[\lim_{\theta \to 0} \frac{\sin\theta}{\theta} \right] \cdot \frac{\lim_{\theta \to 0} \sin\theta}{\lim_{\theta \to 0} \cos\theta + 1}$$

$$= 1$$

$$= -\frac{0}{1+1} = -\frac{0}{2}$$

$$= 0 \quad \#$$

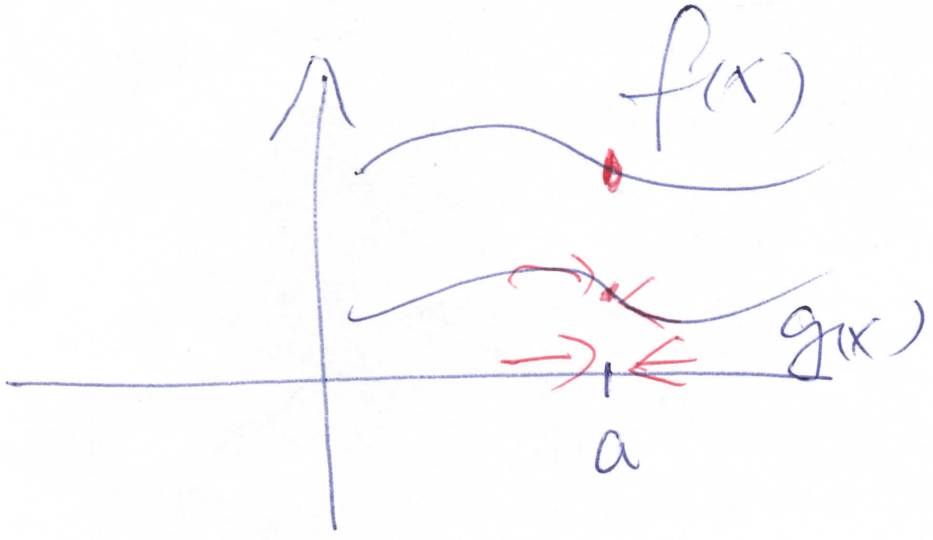
4: Theorem: If $f(x) \geq g(x)$

when $x \to$ near "a"

and both limits:

$\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

Then: $\lim_{x \to a} f(x) \geq \lim_{x \to a} g(x)$.



5: The Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ when x is near "a".

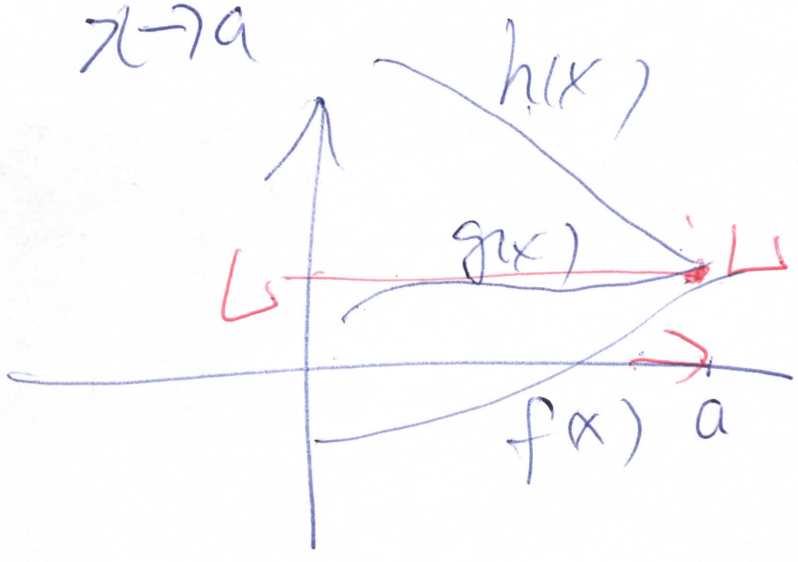
($f(x)$: lower bound; $h(x)$: upper bound)

~~then~~ and:

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L.$$

Then:

$$\lim_{x \to a} g(x) = L.$$



Ex $\lim_{x\to 0} x^2 \sin\frac{1}{x}$ — $g(x)$

Solution:

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$$

$$\leftarrow -x^2 \le x^2 \sin\frac{1}{x} \le x^2 \rightarrow 0$$

Because: $\lim_{x\to 0} x^2 = 0$.

$$\lim_{x\to 0} (-x^2) = 0$$

Squeez Thm $\Rightarrow \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$ #

§1.5 Continuity

1: Definition: A function $f(x)$ is continuous at "a" if

$$\lim_{x \to a} f(x) = f(a) \Rightarrow \text{D.S.P.}$$

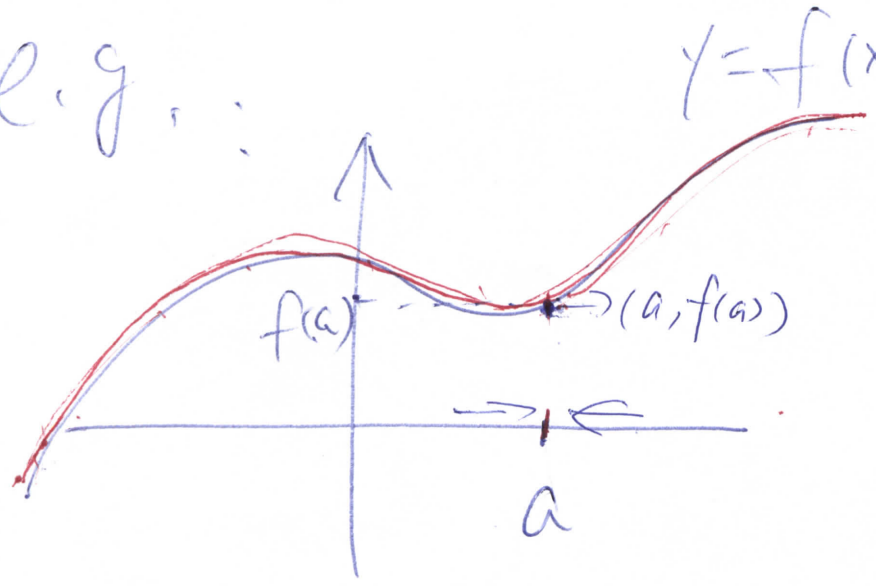
$\Uparrow$ Poly. Rational.

Remarks: ① If a is in the domain of $f(x)$, $f(a)$ is defined. then $f(x)$ is possibly continuous.

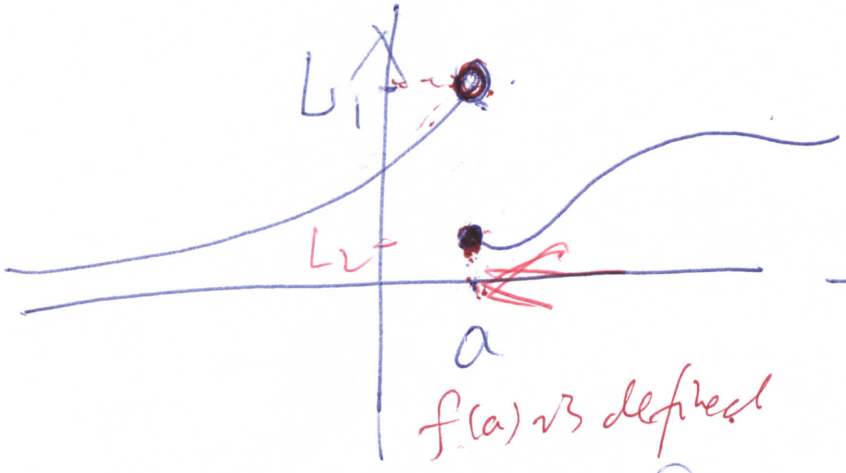
② $\lim_{x \to a} f(x)$ exists.

③ Make sure they are equal to each other.

e.g.:



continuity at "a"



discontinuous function

because: f(a) is defined

$\lim_{x \to a^-} f(x) = L_1$

$\lim_{x \to a^+} f(x) = L_2$

$L_1 \neq L_2$

$\Rightarrow \lim_{x \to a} f(x)$ does not exist

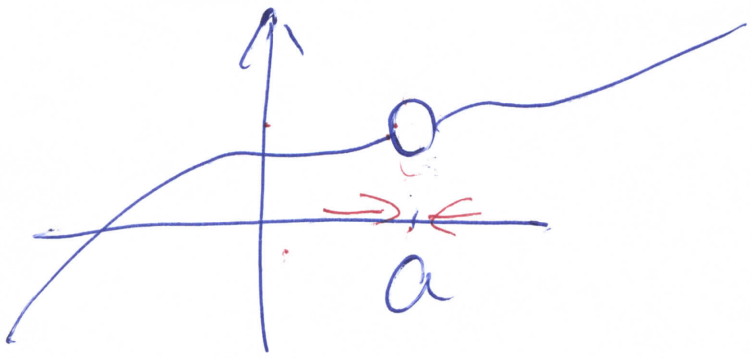
2: Discontinuity.

① Jump type discontinuity.

in which $\lim_{x \to a} f(x)$ does not exist.

or $\lim_{x \to a} f(x) \neq f(a)$

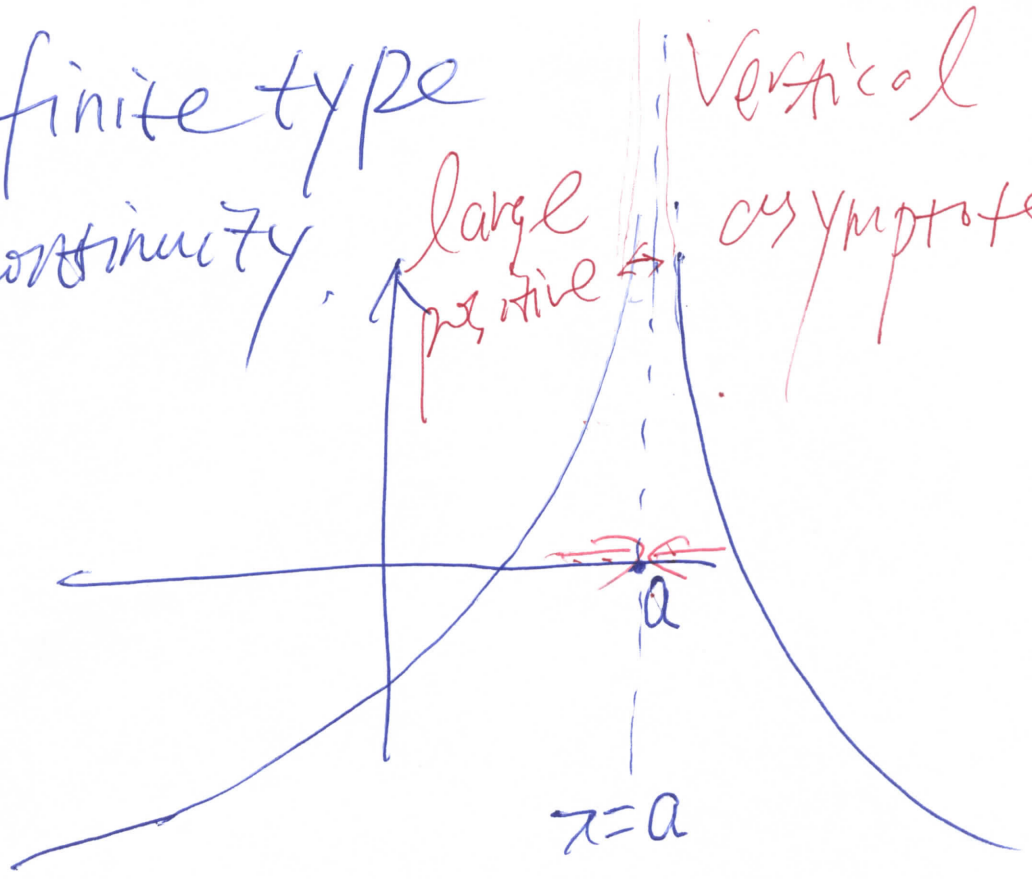
② Removable type discontinuity.



$\lim_{x \to a} f(x)$ exist.

$f(a)$ is undefined.

(3) Infinite type discontinuity.

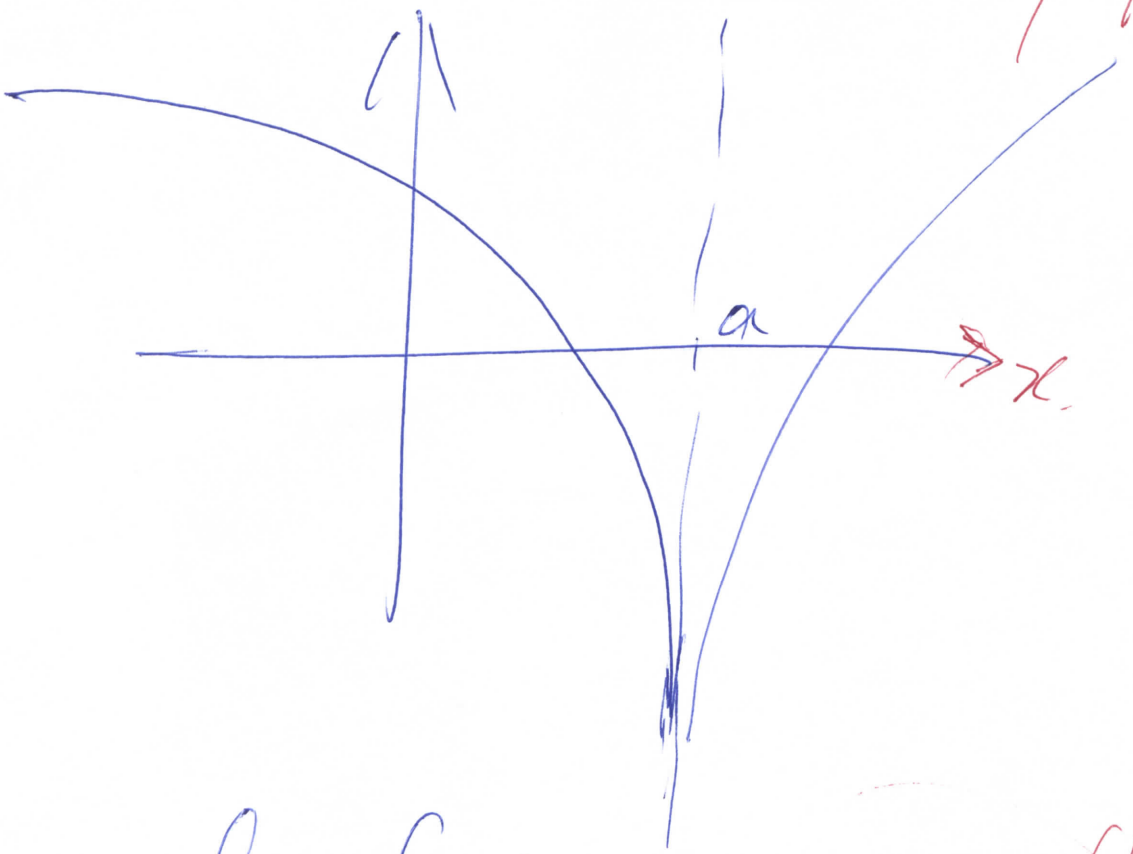


$$\lim_{x \to a} f(x) = +\infty$$ (large positive)

which also means $\lim_{x \to a} f(x)$ does not exist

"∞": is NOT a number

Vertical asymptote.



$\lim_{x \to a} f(x) = -\infty$ (large negative)

Ex 1: Where are each function discontinuous?

(a) $f(x) = \dfrac{x^2+x-2}{x-2}$.

If $x-2=0 \Rightarrow x=2, \Rightarrow f(x)$ undefined at "2" ~~and~~

$\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$ DNE.

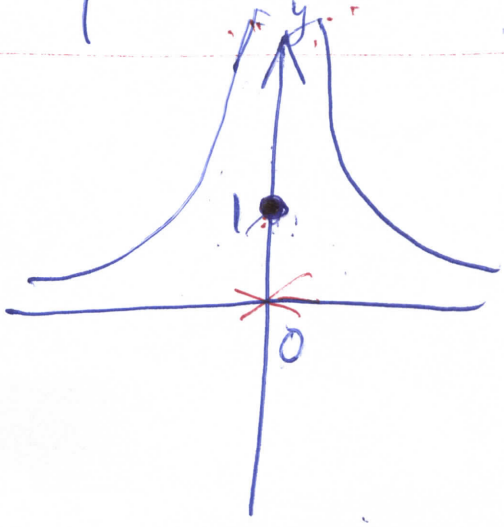
$\Rightarrow$ Removable type discontinuity

and discontinuous at $x = 2$,

because $f(2)$ is undefined

(b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \Rightarrow f(0) = 1 \end{cases}$

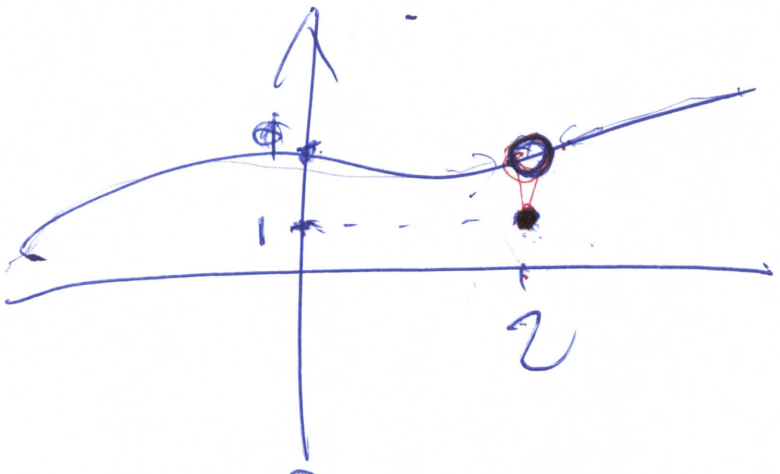
$\lim_{x \to 0} \frac{1}{x^2} = \infty$



$\Rightarrow$ Infinite type.

discontinuous at $x = 0$.

(c) $f(x) = \begin{cases} \frac{x^2-x-3}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases} \Rightarrow f(2) = 1$



$\lim_{x \to 2} f(x) \neq f(2) = 1$

$\Rightarrow$ Jump type

discontinuous at $x = 2$.

eg Examples of continuous functions:
All polynomials, rational functions ....
sin, cos, tan, cot, sec, csc.
are continuous on their domain.

One-sided Continuity

① Continuity at "a" from the right

$$\text{if: } \lim_{x \to a^+} f(x) = f(a)$$

② Continuity at "a" from the left

$$\text{if: } \lim_{x \to a^-} f(x) = f(a)$$

e.g.: $[|x|]$ is continuous from the right at each integer and "0". because:

$$\lim_{x \to n^+} [|x|] = [|n|] = n.$$

The ladder function $\Longleftrightarrow$

the greatest integer function

$$\Big(\; y = [\![x]\!] \;\Big), \quad x \in (-\infty, \infty)$$

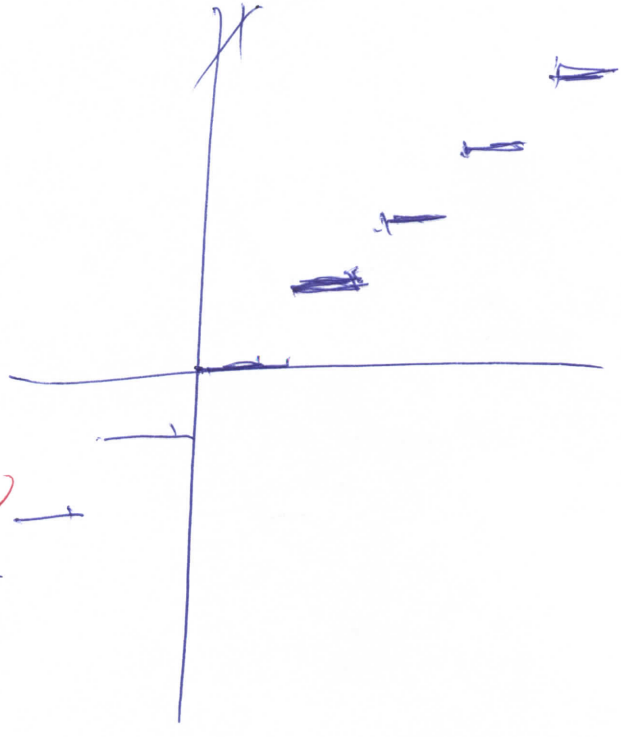
which is the largest integer but less than x.

e.g. $[\![2.1]\!] = 2$

$[\![-2.1]\!] = -3$

$[\![-100]\!] = -100$

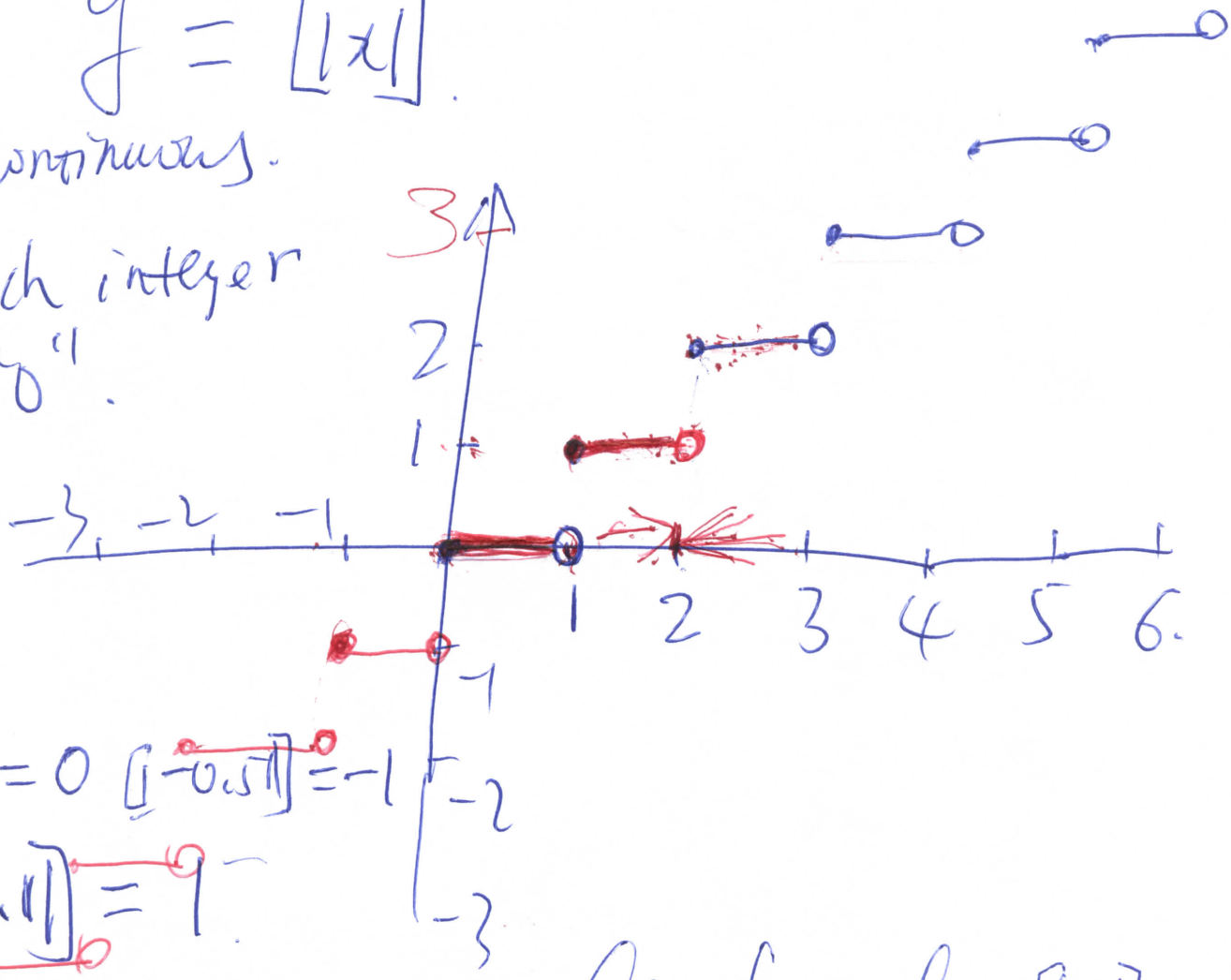
$[\![100]\!] = 100$.



Ex2: The ladder function

$$y = [|x|]$$

→ is discontinuous at each integer and "0".



$[|0.5|] = 0$ $[|-0.5|] = -1$

$[|1.1|] = 1$

$[|-1.1|] = -2$.

$[|2.61|] = 2$.

$[|-2.61|] = -3$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} [|x|]$

$\lim_{x \to 2^-} [|x|] = 1$

$\lim_{x \to 2^+} [|x|] = 2$

} $\lim_{x \to 2} [|x|]$ does not exist

and $[|2|] = 2$.

$[|x|]$ is continuous from the right at each integer and "0"

4: $f(x)$ is Continuous function in an interval if it is continuous at every number in that interval.

Domain

$\Leftrightarrow$ Pick up any $c \in [a,b]$

Such that

$$\lim_{x \to c} f(x) = f(c)$$

#

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2. \end{cases}$$

↓

Continuous.

$$\lim_{x \to 2} f(x) \neq f(2) = 2^3 - 2c$$

$$= 8 - 2c$$

$$\Downarrow$$

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (cx^2 + 2x) = 4c + 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^3 - cx) = 8 - 2c$$

Let $4c + 4 = 8 - 2c$

$$\Downarrow$$

$$6c = 4 \Rightarrow c = \frac{4}{6} = \boxed{\frac{2}{3}} \quad \#$$

5: Theorem of continuity:

If f and g are continuous functions at "a", then the following functions are also continuous at "a"

① $f+g$ ② $f-g$ ③ cf or cg

④ fg ⑤ $\frac{f}{g}$ if $g \neq 0$.

e.g.: $\sqrt{x}$ is continuous in $[0, \infty)$

$\sqrt[3]{x}$ is continuous on $(-\infty, \infty)$

$\sqrt{x}$ is continuous in $[0, \infty)$

Ex2: On what interval is the function continuous? → Domain

$$h(x) = \sqrt{x} + \frac{x+1}{x-1} - \frac{x+1}{x^2+1}.$$

Solution: The domain of $\sqrt{x}$: $[0, \infty)$

------ of $\frac{x+1}{x-1}$: $(-\infty, 1) \cup (1, \infty)$

------ of $\frac{x+1}{x^2+1}$: $(-\infty, \infty)$

$\Rightarrow$ The domain of $h(x)$: $[0, 1) \cup (1, \infty)$

in which $h(x)$ is continuous.

§: Theorem: $f \circ g$ is continuous at "a" if $g(x)$ is continuous at "a" and $f(x)$ is continuous at $g(a)$. $(f \circ g)(x) = f(g(x))$

i.e.: $\lim_{x \to a} (f \circ g)(x) = f(g(a))$

Remark: Direct Substitution Property (D.S.P.) is only a special case of the continuous functions' limit.

Ex3: $\lim_{x \to 1} \frac{1}{\sqrt{x^2+7}-4}$

Solution: The domain of $\frac{1}{\sqrt{x^2+7}-4}$.

$\Leftrightarrow$: $\sqrt{x^2+7}-4 \neq 0$ AND $x^2+7 \geq 0$

$\Downarrow$

$(\sqrt{x^2+7})^2 \neq (4)^2$

$\Downarrow$

$x^2+7 \neq 16$

$\Downarrow$

$\sqrt{x^2} \neq \sqrt{9}$

$\Downarrow$

$x \neq \pm 3 \Rightarrow (-\infty,-3) \cup (-3,3) \cup (3,\infty)$

Then, $\frac{1}{\sqrt{(x^2+7)}-4}$ is continuous

on $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

and, $\lim_{x\to 1} \frac{1}{\sqrt{(x^2+7)}-4} = \frac{1}{\sqrt{(1+7)}-4}$

$$= \frac{1}{\sqrt{8}-4}$$

$$= \frac{1}{2\sqrt{2}-4}. \quad \#$$

7: The intermediate value theorem.
(I.V.T.)

Domain: $a \le x \le b$

$\Downarrow$ $\Downarrow$

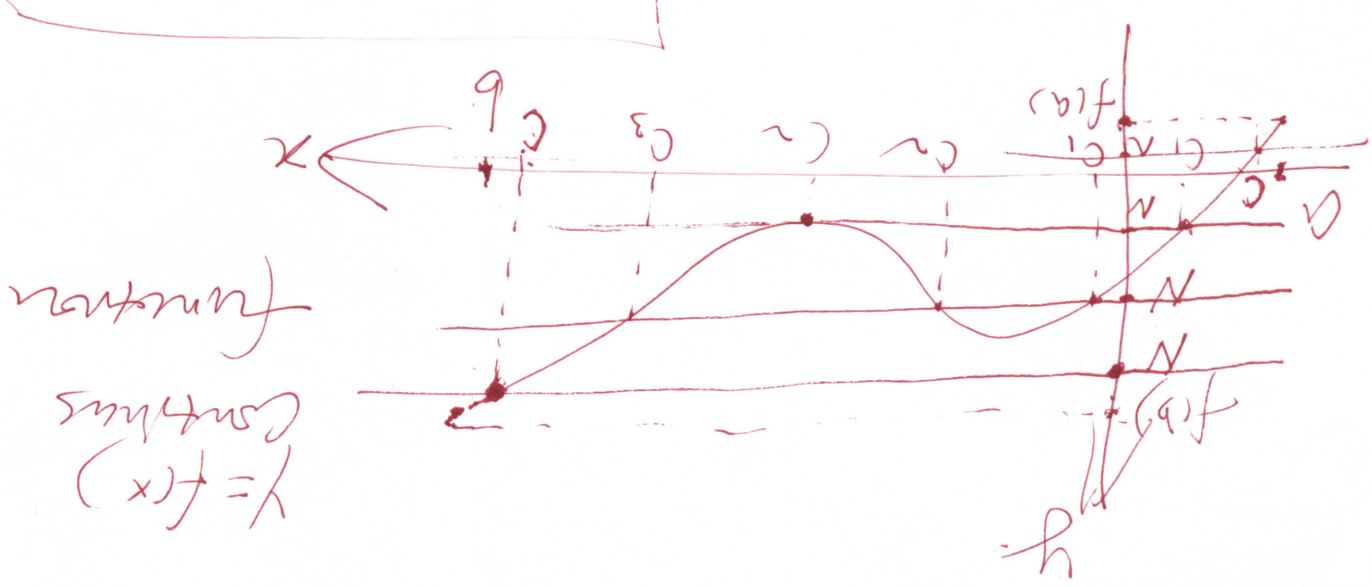
$f(a)$ $f(b)$.

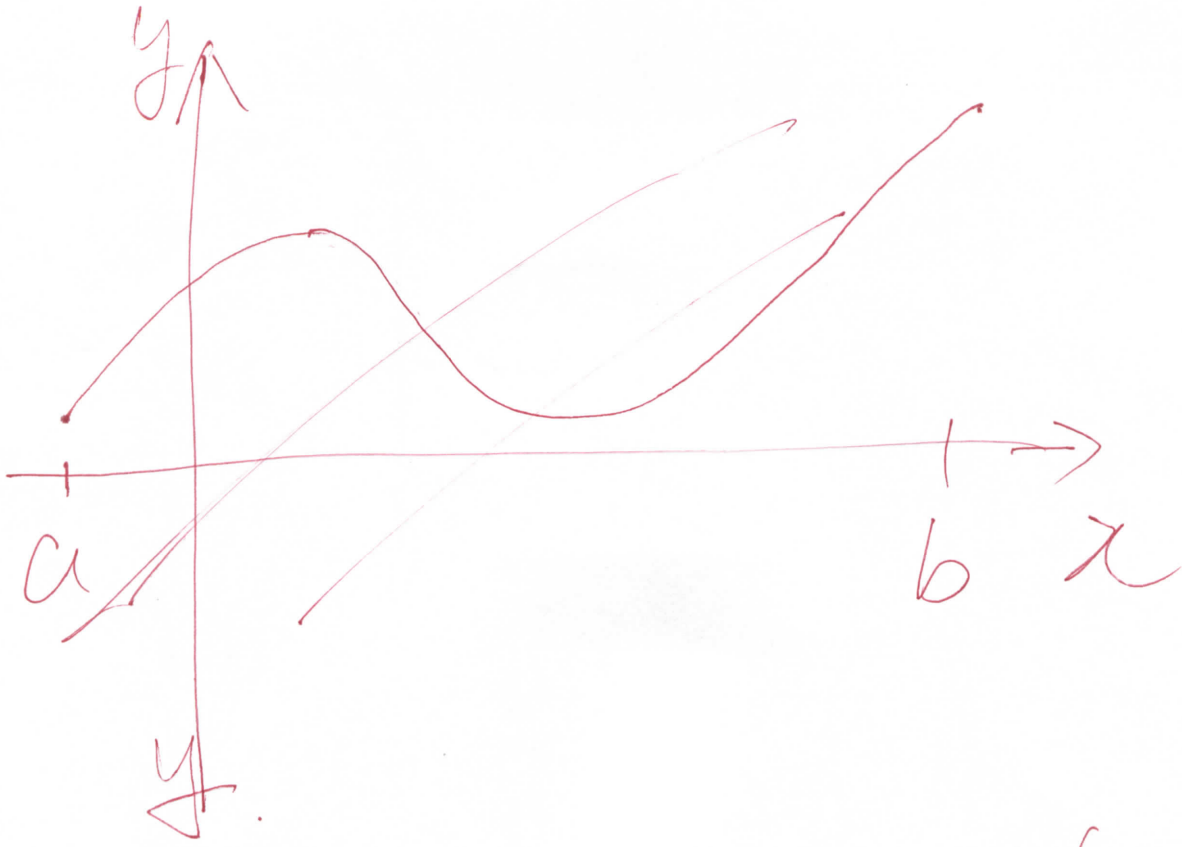
Pick a number $f(a) \le N \le f(b)$

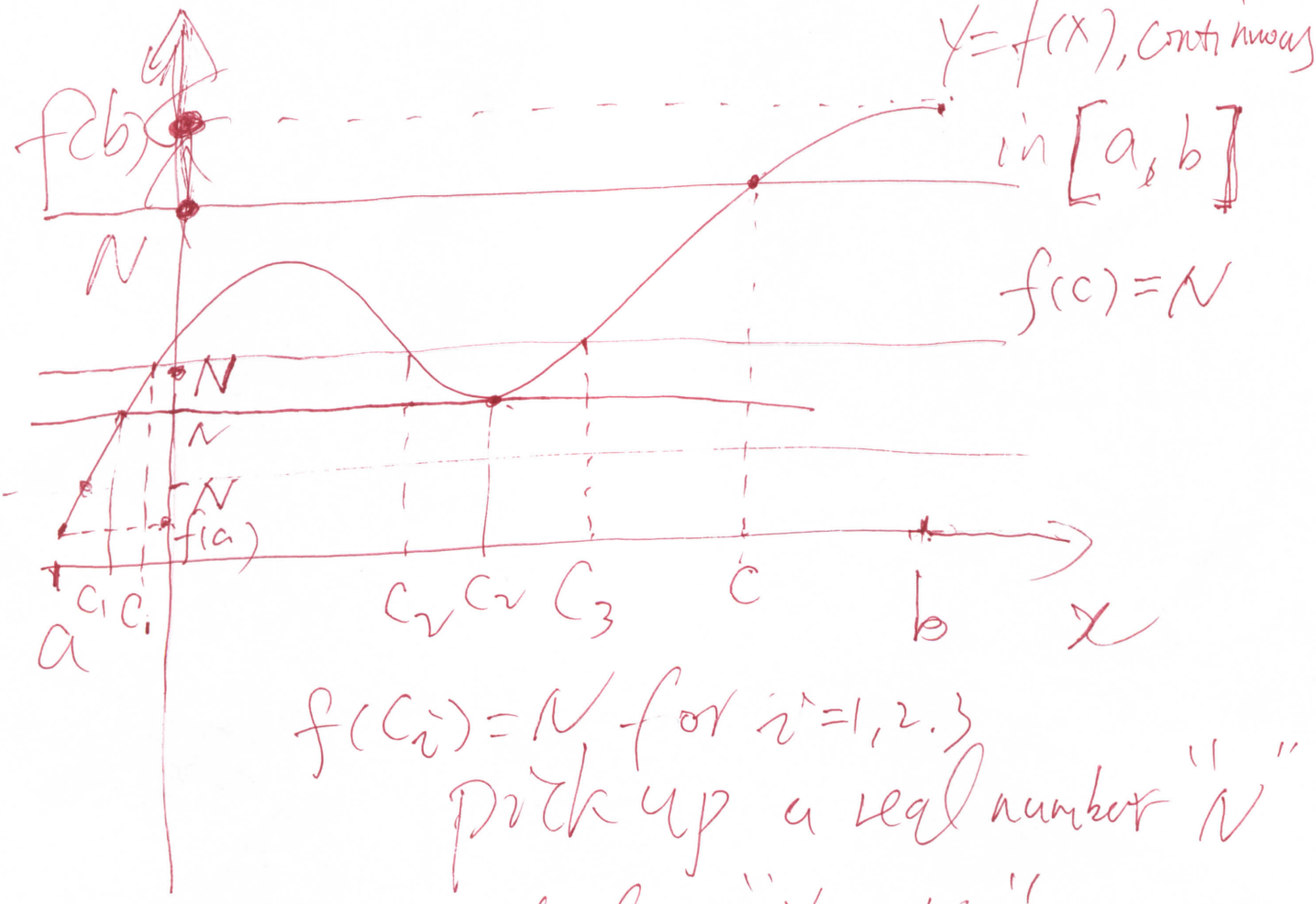
Draw a horizontal line $y = N$

where $f(a) < N < f(b)$

$y = N$







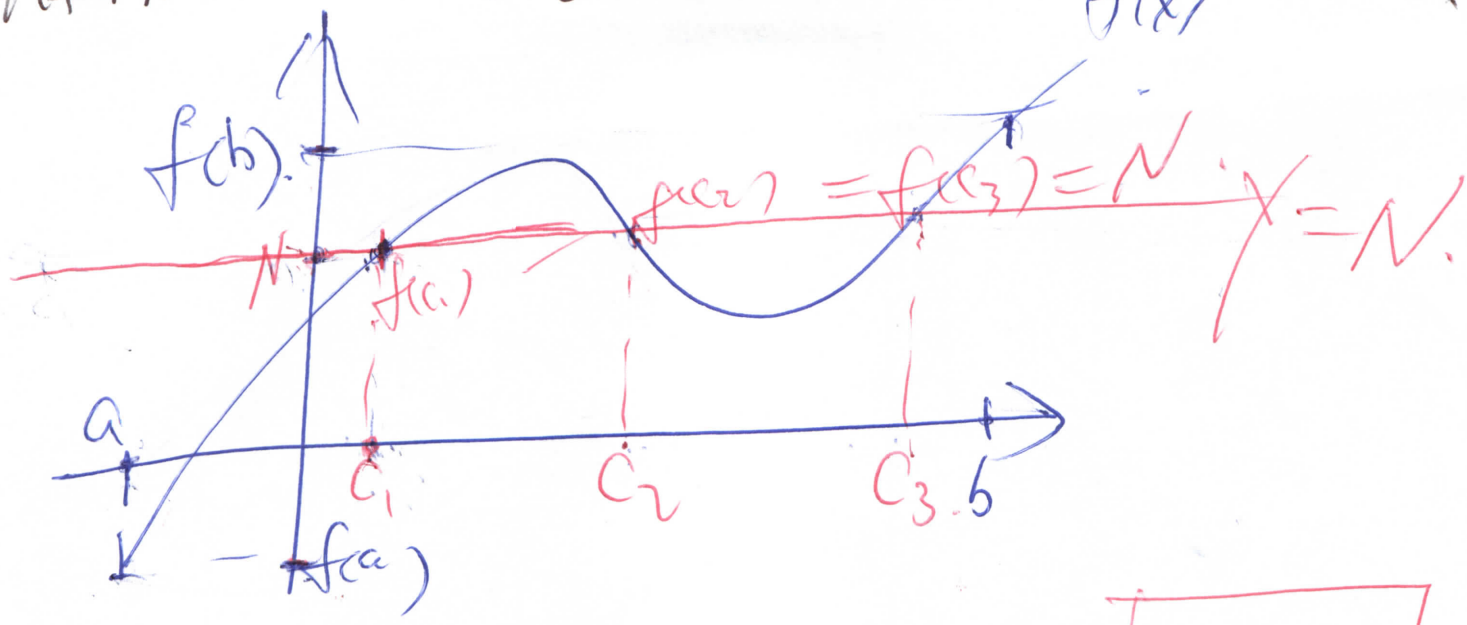
$f(c_i) = N$ for $i = 1, 2, 3$

pick up a real number "N"

and let "$y = N$"

then, $f(a) < N < f(b)$

Intermediate Value Theorem (I.V.T.)



$y = f(x)$ is continuous on $[a, b]$

and let N be any number between $f(a)$ and $f(b)$.

Then: there must exist at least one number $c \in [a, b]$ s.t.

$$f(c) = N.$$

Ex: Show that "Proof!" there is a root of the equation:

$$\underbrace{4x^3-6x^2+3x-2}_{f(x)} = 0 \Rightarrow N=a$$

between 1 and 2

Solution: @ Let $a=1$, $b=2 \Rightarrow [a,b]$ [1,2]

As form a ~~cal~~ closed interval $[a,b]=[1,2]$.

in which the function

$f(x)=4x^3-6x^2+3x-2$ is continuous. Thus by I.V.T. we can always choose N.

Choose: "N = ∞"

between $f(1)$ and $f(2)$

$\|$ $\|$

$4\cdot 1-6\cdot 1+3\cdot 1-2$ $\quad$ $4\cdot 2^3-6\cdot 2^2+3\cdot 2-2$

$\|$ $\|$

$-1<0$ $\quad$ $12>0$

Then we choose $N=0$ that

is between $f(1)=-1$ and $f(2)=12$

By I.V.T. $\Longrightarrow$ there must exist at least

one number $c \in [1,2]$, s.t. (such that)

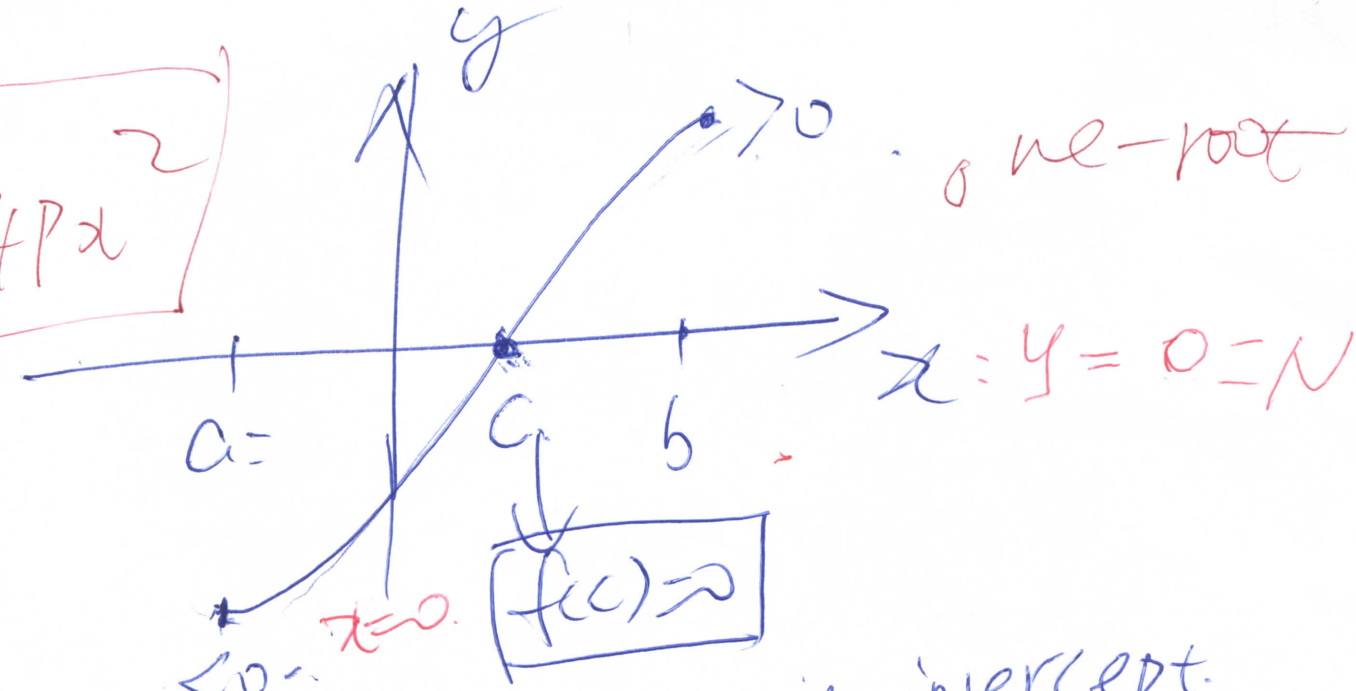
$f(c)=N=0$

$\Downarrow$

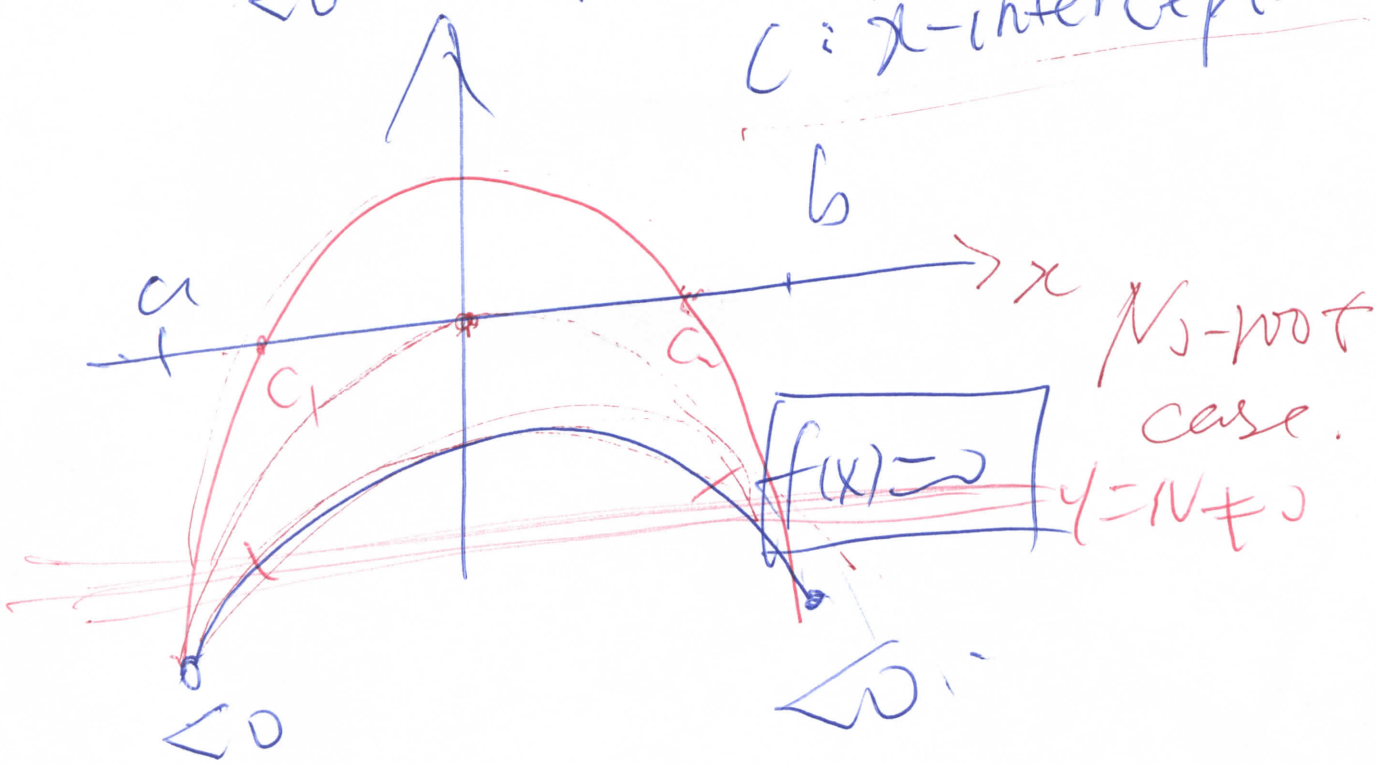
$4c^3-6c^2+3c-2=0$

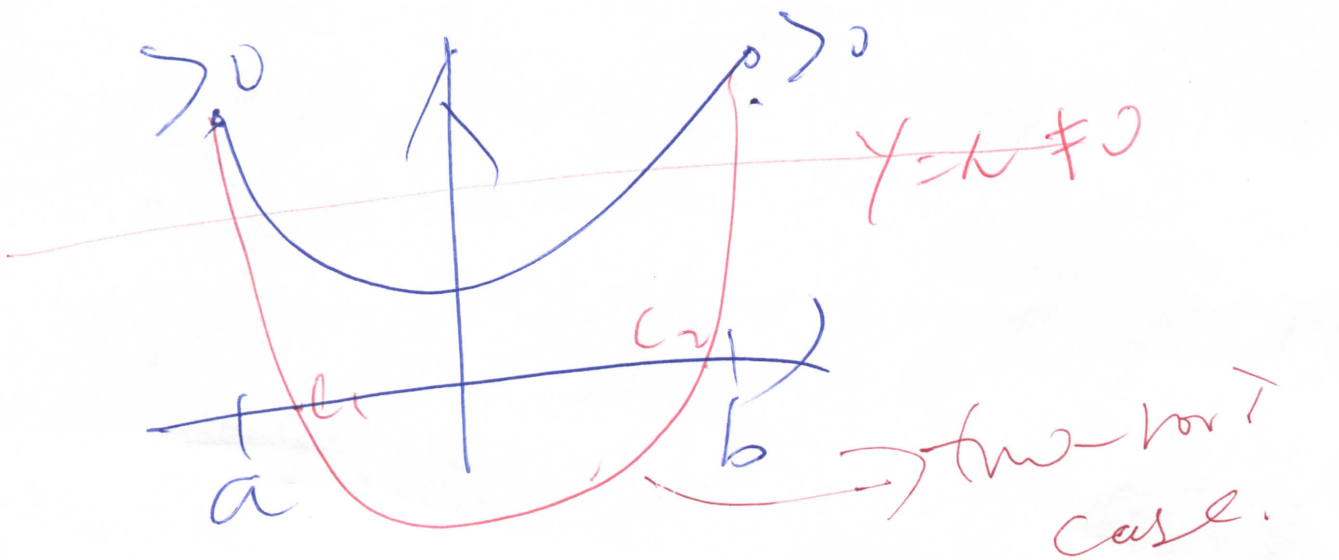
which means: "c" is a root of

$y = 4px^2$



c : x-intercept

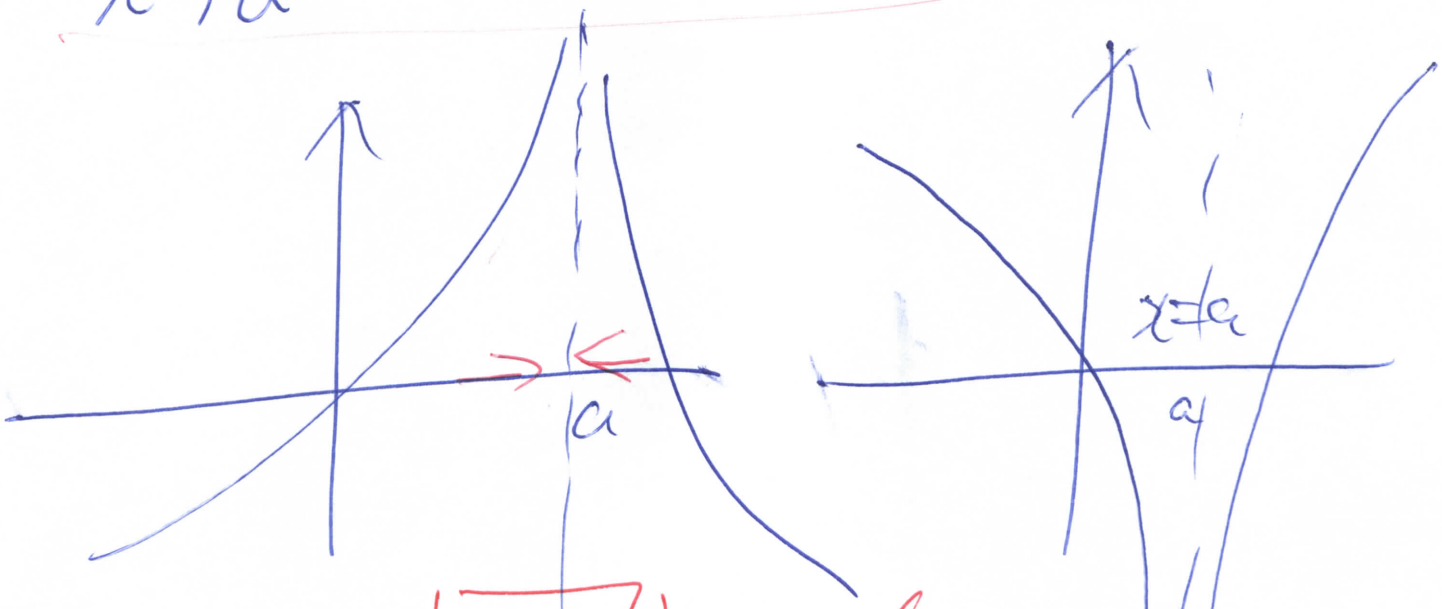




§1.6 Limit involving infinity "∞"

1: Infinite limits

$$\lim_{x \to a} f(x) = \pm\infty$$ $\Rightarrow$ No existing the limit



$x=a$: Vertical asymptote

Remarks: ① "∞" is NOT a number.

"$-\infty$": large negative.

"Large positive"

② This type of limit does NOT exist.

"∞" infinity

① $+\infty + \infty = \infty$

② $-\infty - \infty = -\infty$

③ $\infty \cdot \infty = \infty$ (∞^2)

④ $(-\infty)(-\infty) = (-1)^2 \infty \cdot \infty = \infty$

⑤ $\infty \cdot (-\infty) = -\infty$

⑥ $\infty^n = \infty$ if $n > 0$

e.g. $n = 3 \Rightarrow \infty^3 = \infty$

$n = \frac{1}{2} \Rightarrow \sqrt{\infty} = \infty$

$n = \frac{1}{4} \Rightarrow \sqrt[4]{\infty} = \infty$

(7) $\infty - \infty \neq 0$

indeterminate forms.

1^0 1^∞ 0^0

(8) $\frac{\infty}{\infty} \neq 1$

$\frac{1^{\infty}}{1^{\infty}}$ indeterminate form

③ "$+\infty$" : large positive.

We can read $\lim_{x \to a} f(x) = +\infty$:

as $x \to a$ sufficiently,

$f(x)$ becomes large positive,

arbitrarily.

④ "$-\infty$" : large negative

⑤ Vertical asymptote

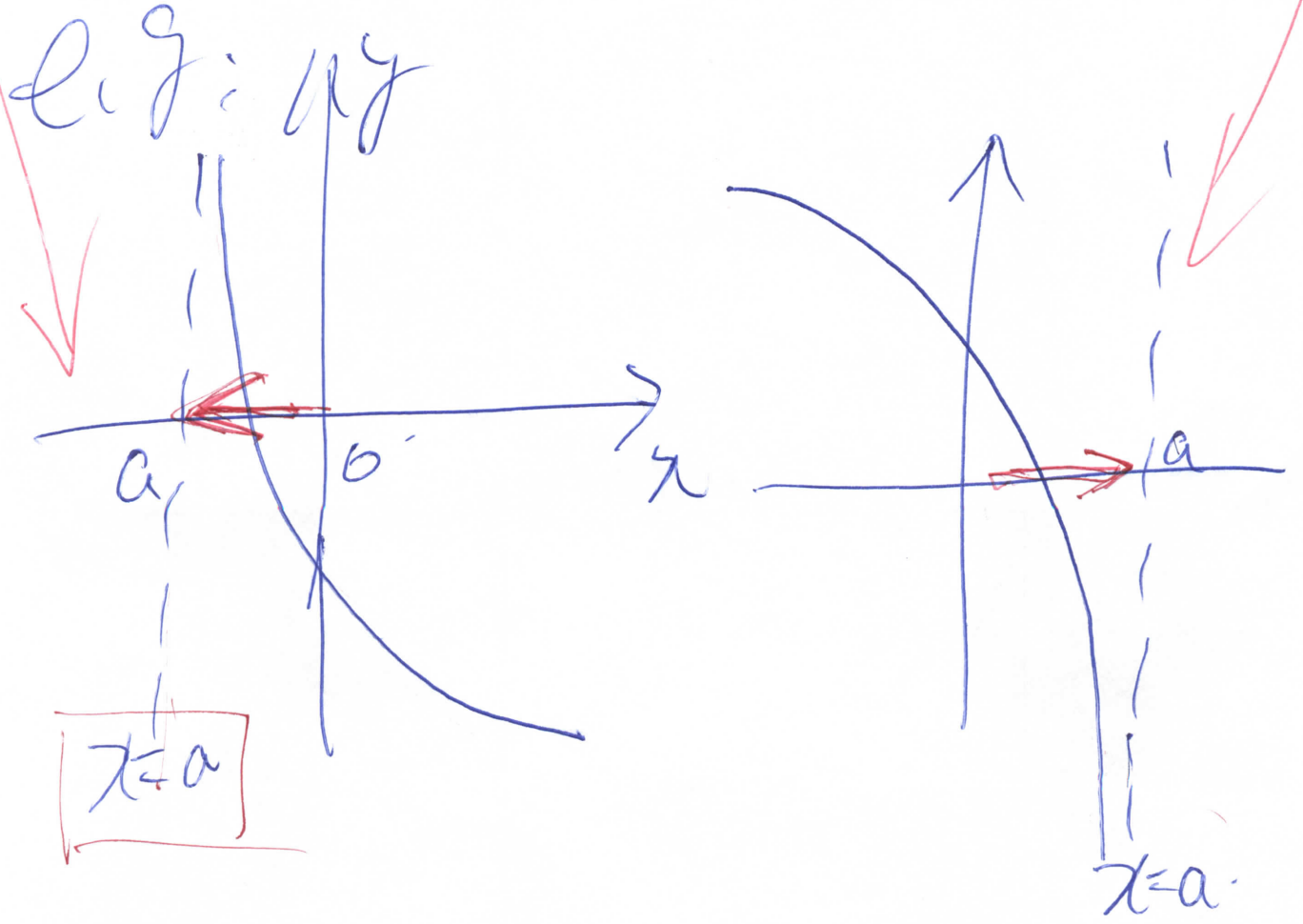
"$x = a$", if ~~∞~~ one of the

following limit exists:

$$\lim_{x \to a^-} f(x) = \infty \quad , \quad \lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty \quad , \quad \lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = \infty \quad , \quad \lim_{x \to a} f(x) = -\infty$$



The fraction : $\frac{a}{b}$

① If a is finite and $a \neq 0$

and b is finite and $b \neq 0$

$\Rightarrow \frac{a}{b}$ is finite.

② If a is finite and $a \neq 0$

and $b = 0$.

$\Rightarrow \frac{a}{0}$ is infinite $= \pm\infty$

$\frac{a}{0^+} \Rightarrow \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0. \end{cases}$

or $\frac{a}{0^-} \Rightarrow \begin{cases} -\infty & \text{if } a > 0 \\ \infty & \text{if } a < 0. \end{cases}$

③ If both a and b are "0"

$\Rightarrow \frac{0}{0} \Rightarrow$ indeterminate form.

Ex1: Find $\lim_{x \to 3^+} \frac{2x}{x-3}$

Solution: $f(x) = \frac{2x}{x-3}$ is continuous in its domain $(-\infty, 3) \cup (3, \infty)$

$$\Rightarrow \lim_{x \to 3^+} \frac{2x}{x-3} = \frac{\lim_{x \to 3^+} 2x}{\lim_{x \to 3^+} (x-3)} = \frac{2 \cdot 3}{3-3}$$

$x \to 3^+ \Leftrightarrow x-3 \to 0^+$

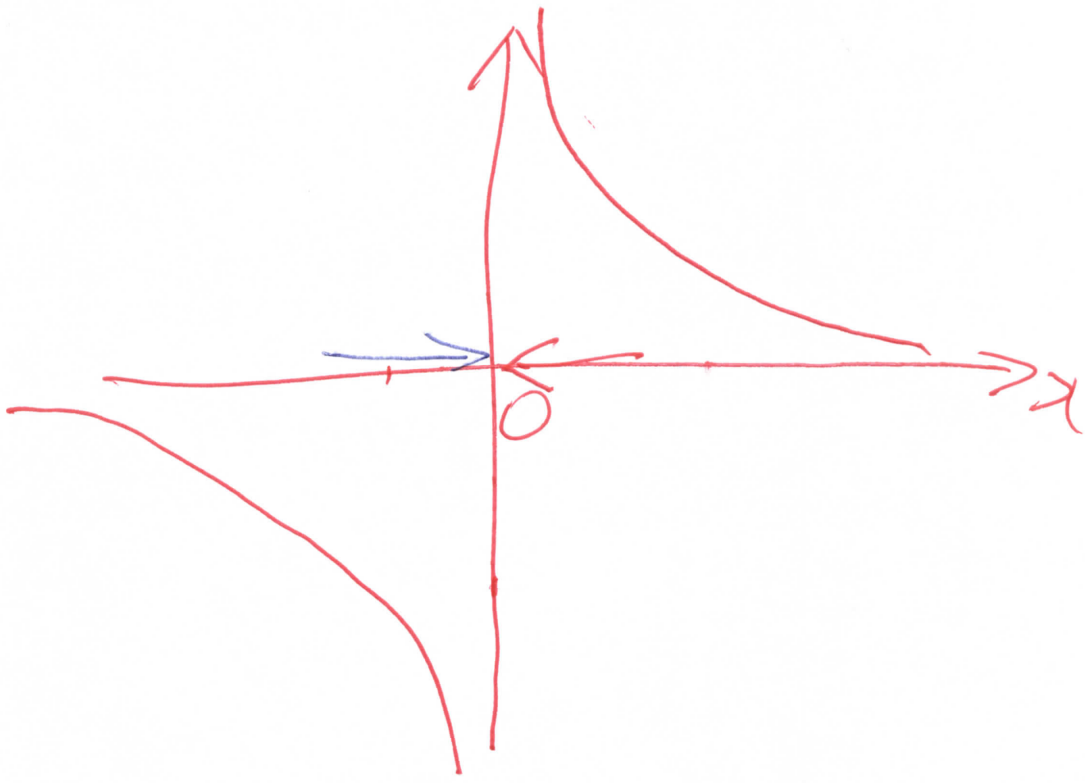
$$= \frac{6}{0^+} \to +\infty$$

$$= \frac{6}{\lim_{x \to 0^+} x} = +\infty \text{ (large positive)}$$

V.A. is: $x = 3$

$\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$



$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 3^-} \frac{2x}{x-3} = \frac{\lim_{x\to 3^-} 2x}{\lim_{x\to 3^-}(x-3)}$$

$x\to 3^-$

$$= \frac{6}{\lim_{x\to 3^-}(x-3)} \quad (6 \to \text{finite number})$$

$$= -\infty$$

$x\to 3^- \Rightarrow (x-3)\to 0$

V.A. $x=3$.

(large negative)

$3 - 0.1 = 2.9$

$2.9 \to 3^-$

$2.99 \to 3^-$

$2.999 \to 3^-$

$= -0.0001 \to 0^-$

$x \to 3^-$

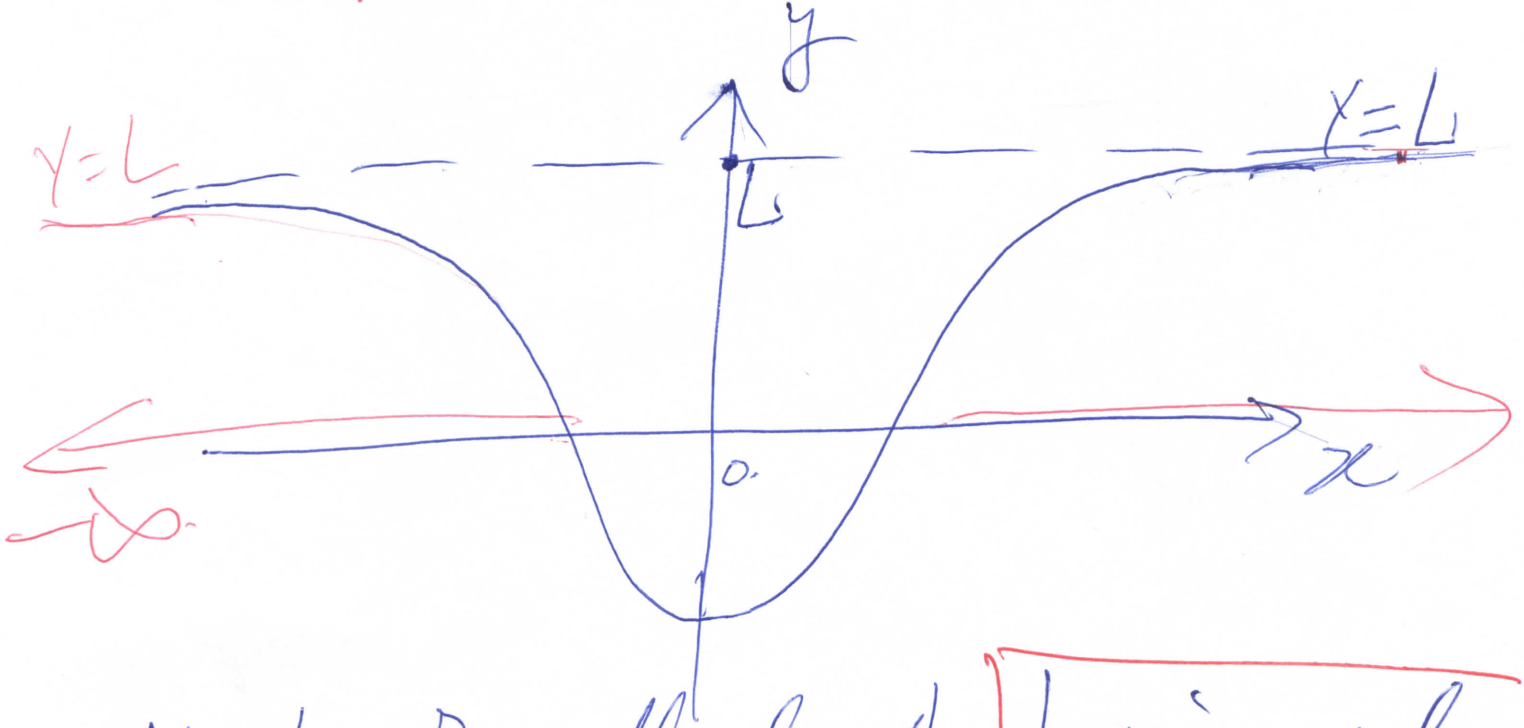
$x-3 \to 3^- - 3 = 0^-$

$\Downarrow$

$x-3 \to 0^-$

2: Limit at infinity.

$$\lim_{x \to \pm\infty} f(x) = L \quad \text{(finite number)}$$



$y = L$ is called the **horizontal asymptote** if one of the following limits holds:

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

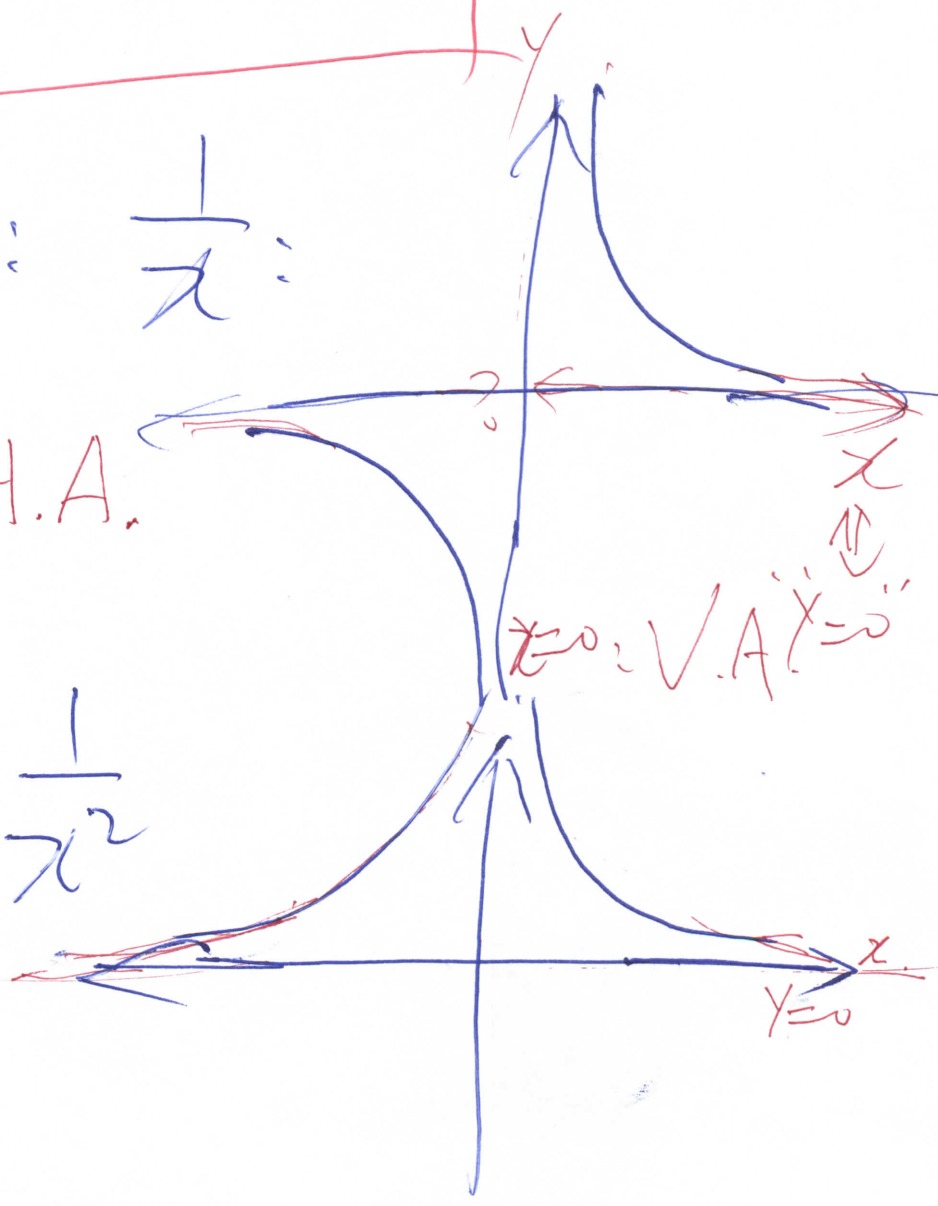
$$\text{or} \quad \lim_{x \to \pm\infty} f(x) = L$$

3: If n is a positive integer

Then: $\lim_{x \to \pm\infty} \frac{1}{x^n} = 0$ $(n \geq 1)$

e.g.: $n=1$: $\frac{1}{x}$:

x axis $(Y=0)$ is H.A.



$n=2$: $\frac{1}{x^2}$

4: Limit of rational functions at infinity:

Ex 3: $\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$

Solution:

Step 1: Pick up the highest order term of "x" in the denominator.

Step 2: then divide both by this term.

$$= \lim_{x \to \infty} \frac{\frac{3x^2 - x - 2}{5x^2}}{\frac{5x^2 + 4x + 1}{5x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{5} - \frac{1}{5x} - \frac{2}{5x^2}}{\boxed{1} + \frac{4}{5x} + \frac{1}{5x^2}}$$

"crucial"

$$= \frac{\frac{3}{5} - \frac{1}{5}\lim_{x\to\infty}\frac{1}{x} - \frac{2}{5}\lim_{x\to\infty}\frac{1}{x^2}}{1 + \frac{4}{5}\lim_{x\to\infty}\frac{1}{x} + \frac{1}{5}\lim_{x\to\infty}\frac{1}{x^2}}$$

$$= \boxed{\frac{3}{5}} \quad \#$$

$$\boxed{\lim_{x\to\infty}\frac{1}{x} = \lim_{x\to\infty}\frac{1}{x^2} = 0}$$

$$= \frac{\frac{3}{5} - \frac{1}{5}\cdot 0 - \frac{2}{5}\cdot 0}{1 + \frac{4}{5}\cdot 0 + \frac{1}{5}\cdot 0}$$

$$= \frac{3}{5}$$

$$\lim_{x\to\infty} \frac{3x-2}{5x^2+4x+1}$$

$$= \lim_{x\to\infty} \frac{\frac{3x-2}{5x^2}}{\frac{5x^2+4x+1}{5x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3}{5}\frac{1}{x} - \frac{2}{5}\frac{1}{x^2}}{1+\frac{4}{5}\frac{1}{x}+\frac{1}{5}\frac{1}{x^2}}$$

$$= \frac{\frac{3}{5}\lim_{x\to\infty}\frac{1}{x} - \frac{2}{5}\lim_{x\to\infty}\frac{1}{x^2}}{1+\frac{4}{5}\lim_{x\to\infty}\frac{1}{x}+\frac{1}{5}\lim_{x\to\infty}\frac{1}{x^2}} = \frac{0-0}{1+0+0}$$

$$= \frac{0}{1} = 0$$

$$\lim_{x\to\infty} \frac{3x^2-x-2}{4x+1}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2-x-2}{4x}}{\frac{4x+1}{4x}} = \lim_{x\to\infty} \frac{\frac{3}{4}x-\frac{1}{4}-\frac{1}{2}\frac{1}{x}}{1+\frac{1}{4}\frac{1}{x}}$$

$$= \frac{\frac{3}{4}\lim_{x\to\infty} x - \frac{1}{4} - \frac{1}{2}\lim_{x\to\infty}\frac{1}{x}}{1+\frac{1}{4}\lim_{x\to\infty}\frac{1}{x}} = \frac{\frac{3}{4}\cdot\infty - \frac{1}{4} - \frac{1}{2}\cdot 0}{1+\frac{1}{4}\cdot 0}$$

$$= \frac{\infty - \frac{1}{4}}{1} = \frac{\infty}{1} = \infty$$

Summary: (Remark)

① For rational function whose numerator and denominator have the same degree.

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = L \neq 0$$

② If $\deg(P(x)) < \deg(Q(x))$

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = 0$$

③ 若 $\deg(P(x)) > \deg(Q(x))$

$$\lim_{x \to b} \frac{P(x)}{Q(x)} = \infty$$

Find the horizontal & vertical asymptotes of the curve:

$$y = \frac{2x^2+x-1}{x^2+x-2}.$$

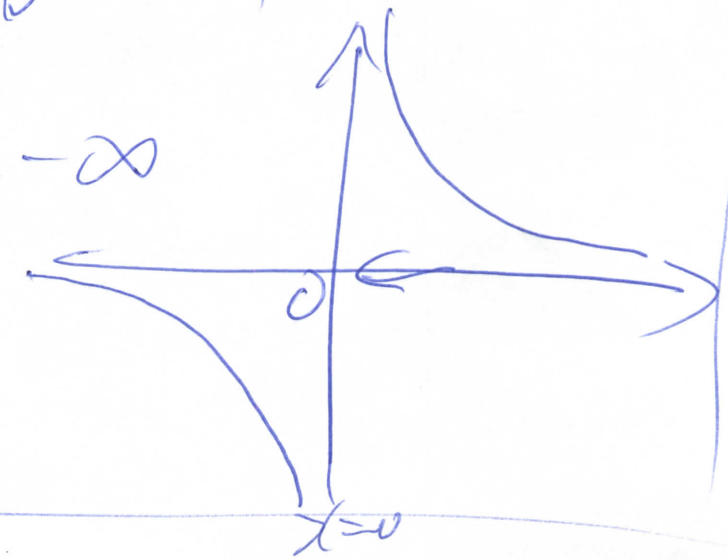
Solution: ① V.A ($x=a$) $\Leftarrow \lim_{x \to [a]^{\pm}} f(x) = \pm[\infty]$

Thus, we need to find out "a" to satisfy:

$$\lim_{x \to "a"} \frac{2x^2+x-1}{x^2+x-2} = \infty.$$

Recall: $\lim_{x \to 0^+} \frac{1}{x} = \infty$.

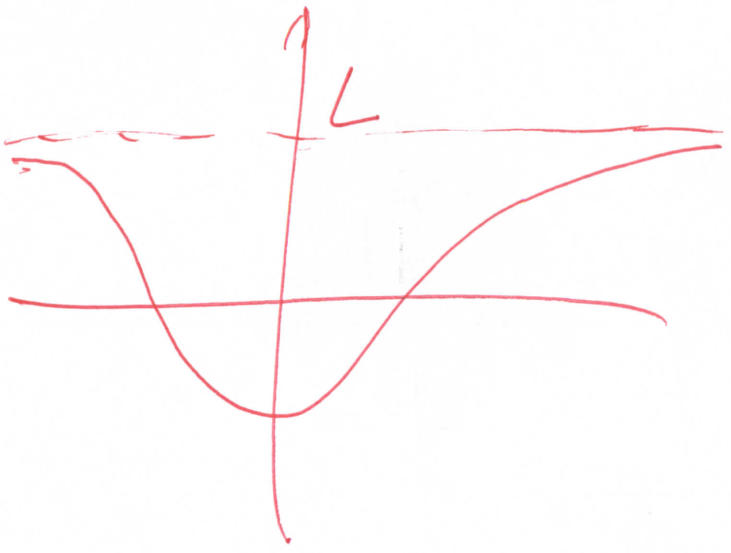
$\lim_{x \to 0^-} \frac{1}{x} = -\infty$



① $\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} = L$

⇓

$y = L$: H.A.



② $\lim_{x\to a^{\pm}} \frac{2x^2+x-1}{x^2+x-2} = \pm\infty$

⇓

$x = a$: V.A.



Similarly: $x=+1$ is another V.A.

② H.A $\Leftarrow \lim f(x) = L$.

"$(y=L)$" $\Rightarrow x \to \pm\infty$

Thus: We evaluate.

$$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2+x-2} \quad \Big/ x^2 \Big/ x^2$$

$$= \lim_{x\to\infty} \frac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}$$

$$= \frac{2}{1} = 2$$

$y=2$ is the only H.A.

$$\lim_{x \to a} \frac{P(x)}{Q(x)} \overset{D.S.P.}{=\!=} \frac{P(a)}{Q(a)}$$

as long as $Q(a) \neq 0$.

$\Rightarrow$ To find "a", we just need to find out at what value of "a" $\frac{P(x)}{Q(x)}$ is not defined!

$\Leftrightarrow$ Find out "a" which makes $Q(x) = 0$.

$$\frac{1}{2} \cdot \frac{\frac{x}{2} - \left(\frac{x}{2}\right) + 1}{\frac{x}{2} - \frac{x}{2} + 2}$$

Let $x^2+x-2=0$. "$\frac{\infty}{\infty}$"

$\Downarrow$

$(x+2)(x-1)=0$

$\Downarrow$

$x+2=0$ or $x-1=0$

$\Downarrow$ $\Downarrow$

$x=-2$ $x=+1$.

When $x \to$ near "-2" or

$x \to$ near "$+1$"

$$\lim_{x\to -2} \frac{2x^2+x-1}{x^2+x-2} \Longrightarrow = \frac{\lim_{x\to -2}(2x^2+x-1)}{\lim_{x\to -2}(x^2+x-2)} =$$

$$= \frac{2\cdot 2^2-2-1}{0} = \frac{5}{0}$$

$= \infty$ $x=-2 \to$ a V.A.

and "$x=+1$" is also a V.A.

Ex: $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$ $\left(\sqrt{x^2+1}+x\right)$

$$= \lim_{x\to\infty} \frac{(x^2+1)-x^2}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x}$$

$$= \frac{1}{\sqrt{\lim_{x\to\infty}(x^2+1)}+\lim_{x\to\infty}x} = \frac{1}{\sqrt{(\lim_{x\to\infty}x)^2+1}+\lim_{x\to\infty}x}$$

$$= \frac{1}{\sqrt{\infty^2+1}+\infty} = \frac{1}{\text{finite}} = \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0$$

$\lim_{x\to\infty}\frac{1}{x}=0$

Ex 5: $\lim_{x\to\infty} \sin x$ does not exist.

because $\sin x$ occillates between "1" and "−1" infinitely often.

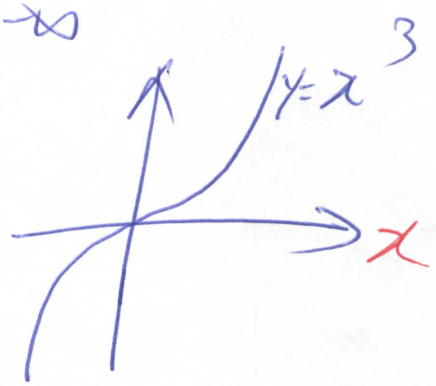
5: **Infinite** limits at **infinity**

$$\lim_{x\to\infty} f(x) = \infty, \quad \lim_{x\to-\infty} f(x) = \infty.$$

$$\lim_{x\to+\infty} f(x) = -\infty, \quad \lim_{x\to-\infty} f(x) = -\infty.$$

e.g. $\lim_{x\to\infty} x^3 = \infty.$

$\lim_{x\to-\infty} x^3 = -\infty.$



$$\frac{1}{\sqrt{\infty^2+1}+\infty} = \frac{1}{\sqrt{\infty+1}+\infty}$$

$$= \frac{1}{\sqrt{\infty}+\infty} = \frac{1}{\infty+\infty}$$

$$= \frac{1}{\infty} = 0$$

∞ : Large positive

$\infty \cdot \infty = \infty^2$: Large positive

$= \infty$

$\sqrt{\infty} = \infty$

$\infty + L = \infty$

$\infty + \infty = \infty$

$\frac{1}{\infty} = 0 \Leftrightarrow \lim_{x \to \infty} \frac{1}{x} = 0$

$\infty - \infty$: No conclusion / indeterminate

$\frac{\infty}{\infty}$: Indeterminate

~~$a^2 - b^2 = (a+b)(a-b)$~~

Summary of ch1:

$$\lim_{x \to a} f(x) = L$$

↑ finite number.

"x" is [near] "a"

High Tech: of Evaluating limits:

① D.S.P.: $\lim_{x \to a} f(x) = f(a)$

if $f(x)$ is polynomial and Rational function in their domain

② Extend D.S.P to all [continuous] function -

③ Laws of limits

11 laws

⑥ Limit involving infinity

④ Theorem of limit

$$\lim_{x \to a} f(x) = L \iff$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L.$$

⑤ If $f(x) \le g(x) \le h(x)$

and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$

$$\Rightarrow \lim_{x \to a} g(x) = L.$$

[Squeeze theorem]